CEE7430 Homework 6 partial solution for questions that caused trouble

3c) Loucks 6.2 c) [Solution from Vinod]

6.2 c) Disaggregated model

$$X_t = A Z_t + B V_t$$

$$A = S_{xz} \, S_{zz}^{-1}$$

$$BB^T = S_{xx} - A S_{zz} A^T$$

we have $Z_t = (x_1 + x_2 + x_3)$

$$S_{zz} = E(Z_t Z_t^T)$$

$$= E\left[(x_t^1 + x_t^2 + x_t^3)(x_t^1 + x_t^2 + x_t^3)\right]$$

$$= E\left[\cancel{x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_1x_2 + x_2x_1 +}\right]$$

$$= \left[x_t^{1^2} + x_t^{2^2} + x_t^{3^2} + x_t^1 x_t^2 + x_t^1 x_t^3 + x_t^2 x_t^1 + x_t^2 x_t^3 + x_t^3 x_t^1 + x_t^3 x_t^2 \right]$$

This Sum of So, which

117.171

$$S_{xz} = E\left[\begin{pmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \end{pmatrix} Z_t\right] = E\left[\begin{pmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \end{pmatrix} (x_t^1 + x_t^2 + x_t^3)\right]$$

$$= E\begin{bmatrix} x_t^{1^2} + x_t^1 x_t^2 + x_t^1 x_t^3 \\ x_t^1 x_t^2 + x_t^{2^2} + x_t^2 x_t^3 \\ x_t^3 x_t^1 + x_t^3 x_t^2 + x_t^{3^2} \end{bmatrix}$$

Sum of Rows of So

$$\begin{pmatrix} 48.056 \\ 53.555 \\ 16.101 \end{pmatrix}$$

$$A = S_{xz} \, S_{zz}^{-1}$$

$$\begin{pmatrix} 48.056 \\ 53.55 \\ 16.1 \end{pmatrix} \frac{1}{117.71} = \begin{pmatrix} 0.408 \\ 0.455 \\ 0.137 \end{pmatrix}$$

$$B.B^T = S_{xx} - A S_{zz} A^T$$

$$= \begin{bmatrix} 20.002 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 2.505 \end{bmatrix} - \begin{pmatrix} .408 \\ .455 \\ .137 \end{pmatrix} \times 117.71 \; (.408 \;\; .455 \;\; .137)$$

$$= \begin{bmatrix} 20.002 & & \\ & & \\ & & 2.505 \end{bmatrix} - \begin{pmatrix} 48.056 \\ 53.55 \\ 16.101 \end{pmatrix} (.408 \;\; .455 \;\; .137)$$

$$= \begin{bmatrix} 20.002 & 21.436 & 6.618 \\ 21.436 & 25.141 & 6.978 \\ 6.618 & 6.978 & 2.505 \end{bmatrix} - \begin{bmatrix} 19.61 & 21.86 & 6.58 \\ 21.84 & 24.365 & 7.336 \\ 6.57 & 7.326 & 2.206 \end{bmatrix}$$

$$= \begin{bmatrix} 0.41 & -0.424 & -0.038 \\ -0.44 & 0.776 & -0.358 \\ 0.048 & -0.348 & 0.299 \end{bmatrix}$$

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BB^T is disaggregation matrix is Singular. Here sum of 1st & 2nd row ~~this~~ looks equal to the third row. In that case the matrix would be singular we have to use the singular value decomposition method to compute B from BB^T.

In R

```
> S0=matrix(c(20.002, 21.436, 6.618, 21.436, 25.141, 6.978, 6.618, 6.978, 2.505), ncol=3)
> S0
     [,1]   [,2]  [,3]
```

```
[1,] 20.002 21.436 6.618
[2,] 21.436 25.141 6.978
[3,]  6.618  6.978 2.505
> S1=matrix(c(6.487, 6.818, 1.638, 7.5, 7.625, 1.815, 2.593, 2.804, 0.6753), ncol=3, byrow=T)
> S1
      [,1]  [,2]   [,3]
[1,] 6.487 6.818 1.6380
[2,] 7.500 7.625 1.8150
[3,] 2.593 2.804 0.6753
> Szz=sum(S0)
> Szz
[1] 117.712
> Sxz=rowSums(S0)
> Sxz
[1] 48.056 53.555 16.101
> A=Sxz/Szz  # This is a scalar operation
> A
          [,1]      [,2]     [,3]
[1,] 0.4082506 0.4549664 0.136783

> BBT=S0-t(A) %*% Szz %*% A
Error: object "BBt" not found
> BBT
            [,1]       [,2]        [,3]
[1,]  0.38310697 -0.4278633  0.04475635
[2,] -0.42786333  0.7752767 -0.34741334
[3,]  0.04475635 -0.3474133  0.30265698
```

Cholesky decomposition

```
> B=chol(BBT)
> B
          [,1]       [,2]          [,3]
[1,] 0.6189564 -0.6912657  7.230939e-02
[2,] 0.0000000  0.5453699 -5.453699e-01
[3,] 0.0000000  0.0000000  1.079641e-07
```

Note - this is upper triangular, so transposed from the sense used in Loucks.
Note also that the 3rd term of the last row is essentially 0, reflecting the singularity of this. There are effectively only two degrees of freedom here. Checking the product.

```
> t(B) %*% B
            [,1]       [,2]        [,3]
[1,]  0.38310697 -0.4278633  0.04475635
[2,] -0.42786333  0.7752767 -0.34741334
[3,]  0.04475635 -0.3474133  0.30265698
```

Note that the original matrix is recovered.

Alternative, using singular value decomposition

```
> svdres=svd(BBT)
> svdres
$d
[1] 1.168492e+00 2.925485e-01 4.119365e-15

$u
          [,1]        [,2]      [,3]
[1,] -0.4633691 -0.67227656 0.5773503
[2,]  0.8138931 -0.06515111 0.5773503
[3,] -0.3505240  0.73742767 0.5773503

$v
          [,1]        [,2]      [,3]
[1,] -0.4633691 -0.67227656 0.5773503
[2,]  0.8138931 -0.06515111 0.5773503
[3,] -0.3505240  0.73742767 0.5773503

> B=svdres$u %*% diag(sqrt(svdres$d))
> B
          [,1]        [,2]         [,3]
[1,] -0.5008872 -0.36361928 3.705566e-08
[2,]  0.8797925 -0.03523877 3.705566e-08
[3,] -0.3789053  0.39885805 3.705566e-08
```

Note that here the 0 values in the last column are due to the singularity.
Checking product.

```
> B %*% t(B)
            [,1]       [,2]        [,3]
[1,]  0.38310697 -0.4278633  0.04475635
[2,] -0.42786333  0.7752767 -0.34741334
[3,]  0.04475635 -0.3474133  0.30265698
```

4. Loucks 6.6

4) Do Loucks et. al 1981 problem 6.6.

6.6 a) $z_{y+1} = AV_y - BV_{y-1}$ — - - - .

multiplying the equation by z_{y+1}^T & taking expectation both sides.

$$\mathcal{E}(z_{y+1} z_{y+1}^T) = \mathcal{E}\left[(AV_y - BV_{y-1}) z_{y+1}^T\right]$$

$$S_0 = \mathcal{E}(AV_y \cdot z_{y+1}^T - BV_{y-1} z_{y+1}^T)$$

$$= A\,\mathcal{E}(V_y z_{y+1}^T) - B\,\mathcal{E}(V_{y-1} z_{y+1}^T) \quad \text{--- (I)}$$

multiplying V_y by z_{y+1}^T and taking exp. both sides

$$\mathcal{E}(V_y z_{y+1}^T) = \mathcal{E}\left[V_y (AV_y - BV_{y-1})^T\right]$$

$$= \mathcal{E}(\underbrace{V_y \cdot V_y^T}_{I} A^T) - \mathcal{E}(\underbrace{V_y V_{y-1}^T}_{0} B^T)$$

$$= A^T \quad \text{— — — — (II)}$$

Multiplying V_{y-1} by z_{y-1}^T & taking exp. both sides.

$$\mathcal{E}(V_{y-1} z_{y-1}^T) = \mathcal{E}(V_{y-1} V_y^T A^T) - \mathcal{E}(V_{y-1} V_{y-1}^T B^T)$$

$$= -B^T$$

Putting A^T & $-B^T$ in eqn 1

$$S_0 = AA^T + BB^T$$

To compute S_1

Multiplying given eq^n by z_y^T & taking expectation

$$\mathcal{E}(z_{y+1} z_y^T) = \mathcal{E}\left[(Av_y - Bv_{y-1}) z_y^T\right]$$

$$= A\,\mathcal{E}(\overset{0}{\cancel{v_y z_y^T}}) - B\,\mathcal{E}(\underbrace{v_{y-1} z_y^T}_{A^T \text{ from (II)}})$$

$$= -BA^T$$

$$S_1 = -BA^T$$

So,
$$S_0 = AA^T + BB^T$$
$$S_1 = -BA^T$$

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6.6 b) AR-2 model

$$z_{y+1} = Az_y + Bz_{y-1} + Cv_y$$

multiplying by z_y^T & taking expectation

$$\mathcal{E}(z_{y+1} z_y^T) = A\,\mathcal{E}(z_y \cdot z_y^T) + B\,\mathcal{E}(z_{y-1} z_y^T) + C\,\mathcal{E}(\overset{0}{\cancel{v_y z_y^T}})$$

$$S_1 = AS_0 + BS_1$$

multiplying by z_{y-1}^T & taking expectation

$$\mathcal{E}(z_{y+1} z_{y-1}^T) = A\,\mathcal{E}(z_y z_{y-1}^T) + B\,\mathcal{E}(z_{y-1} z_{y-1}^T) + \mathcal{E}(\overset{0}{\cancel{v_y v_{y-1}^T}})$$

$S_2 = AS_1 + BS_0$

~~multiply~~

multiplying by z_{y+1}^T & taking expectation

$$E\left(z_{y+1} z_{y+1}^T\right) = E\left[\left(Az_y + Bz_{y-1} + CV_y\right)\left(Az_y + Bz_{y-1} + CV_y\right)^T\right]$$

$$= Az_y z_y^T A^T + Az_y z_{y-1}^T B^T + Az_y \cancel{V_y^T} C^T +$$

$$Bz_{y-1} z_y^T A^T + Bz_{y-1} z_{y-1}^T B^T + B\cancel{z_{y-1} V_y^T} C^T +$$

$$CV_y \cancel{z_y^T} A^T + CV_y \cancel{z_{y-1}^T} B^T + CV_y V_y^T C^T$$

So,

$$S_0 = AS_0A^T + AS_1B^T + BS_1^TA^T + BS_0B^T + CC^T$$

Not symmetric

$$\cancel{CC^T = AS_0A^T + AS_1B^T + BS_1A^T + BS_0B^T}$$

$$CC^T = S_0 - \underline{AS_0A^T} - AS_1B^T - \underline{BS_1A^T} - BS_0B^T$$

$$= S_0 - \left(AS_0 + BS_1\right)A^T - \left(AS_1 + BS_0\right)B^T$$

$$= S_0 - S_1A^T - S_2B^T$$

CC^T can be decomposed to C by cholesky decomposition.

For A & B

we have,

$$A = (S_1 - BS_1)\, S_0^{-1}$$

$$A = (S_2 - BS_0)\, S_1^{-1}$$

So, $(S_1 - BS_1)\, S_0^{-1} = (S_2 - BS_0)\, S_1^{-1}$

$$S_1 S_0^{-1} - BS_1 S_0^{-1} = S_2 S_1^{-1} - BS_0 S_1^{-1}$$

$$B = (S_1 S_0^{-1} - S_2 S_1^{-1})\, (S_1 S_0^{-1} - S_0 S_1^{-1})^{-1}$$

So, $A = (S_1 - BS_1)\, S_0^{-1}$ or $(S_2 - BS_0)\, S_1^{-1}$

$$B = (S_1 S_0^{-1} - S_2 S_1^{-1})\, (S_1 S_0^{-1} - S_0 S_1^{-1})^{-1}$$

$$CC^T = S_0 - S_1 A^T - S_2 B^T.$$

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