Distance from bank (ft)	0	30	60	80	100	120	140	160
Depth (ft)	0	18.5	21.5	22.5	23.0	22.5	22.5	22.0
Velocity (ft/s)	0	0.55	1.70	3.00	3.06	2.91	3.20	3.36

Distance	180	200	220	240	260	280	300	320	340
Depth	22.0	23.0	22.0	22.5	23.0	22.8	21.5	19.2	18.0
Velocity	3.44	2.70	2.61	2.15	1.94	1.67	1.44	1.54	0.81

Distance	360	380	410	450	470	520	570	615
Depth	14.7	12.0	11.4	9.0	5.0	2.6	1.3	0
Velocity	1.10	1.52	1.02	0.60	0.40	0.33	0.29	0

6.3.2 Plot a graph of velocity vs. distance from the bank for the data given in Prob. 6.3.1. Plot a graph of velocity vs. depth of flow.

6.3.3 The observed gage height during a discharge measurement of the Colorado River at Austin is 11.25 ft. If the measured discharge was 9730 ft^3/s, calculate the percent difference between the discharge given by the rating curve (Fig. 6.3.8) and that obtained in this discharge measurement.

6.3.4 The bed slope of the Colorado River at Austin is 0.03 percent. Determine, for the data given in Example 6.3.1, what value of Manning's n would yield the observed discharge for the data shown.

6.3.5 A discharge measurement on the Colorado River at Austin, Texas, on June 16, 1981, yielded the following results. Calculate the discharge in ft^3/s.

Distance from bank (ft)	0	35	55	75	95	115	135	155
Depth (ft)	0	18.0	19.0	21.0	20.5	18.5	18.2	19.5
Velocity (ft/s)	0	0.60	2.00	3.22	3.64	3.74	4.42	3.49

Distance	175	195	215	235	255	275	295
Depth	20.0	21.5	21.5	21.5	22.0	21.5	20.5
Velocity	5.02	4.75	4.92	4.44	3.94	2.93	2.80

Distance	325	355	385	425	465	525	575
Depth	17.0	13.5	10.6	9.0	6.1	2.0	0
Velocity	2.80	1.52	1.72	0.95	0.50	0.39	0

6.3.6 If the bed slope is 0.0003, determine the value of Manning's n that would yield the same discharge as the value you found in Problem 6.3.5.

6.3.7 The observed gage height for the discharge measurement in Prob. 6.3.5 was 19.70 ft above datum. The rating curve at this site is shown in Fig. 6.3.8. Calculate the percent difference between the discharge found from the rating curve for this gage height and the value found in Prob. 6.3.5.

CHAPTER 7

UNIT HYDROGRAPH

In the previous chapters of this book, the physical laws governing the operation of hydrologic systems have been described and working equations developed to determine the flow in atmospheric, subsurface, and surface water systems. The Reynolds transport theorem applied to a control volume provided the mathematical means for consistently expressing the various applicable physical laws. It may be remembered that the control volume principle does not call for a description of the internal dynamics of flow within the control volume; all that is required is knowledge of the inputs and outputs to the control volume and the physical laws regulating their interaction.

In Chap. 1, a tree classification was presented (Fig. 1.4.1), distinguishing the various types of models of hydrologic systems according to the way each deals with the randomness and the space and time variability of the hydrologic processes involved. Up to this point in the book, most of the working equations developed have been for the simplest type of model shown in this diagram, namely a deterministic (no randomness) lumped (one point in space) steady-flow model (flow does not change with time). This chapter takes up the subject of deterministic lumped unsteady flow models; subsequent chapters (8–12) cover a range of models in the classification tree from left to right. Where possible, use is made of knowledge of the governing physical laws of the system. In addition to this, methods drawn from other fields of study such as linear systems analysis, optimization, and applied statistics are employed to analyze the input and output variables of hydrologic systems.

In the development of these models, the concept of control volume remains as it was introduced in Chap. 1: "A volume or structure in space, surrounded by a boundary, which accepts water and other inputs, operates on them internally and produces them as outputs." In this chapter, the interaction between rainfall and runoff on a watershed is analyzed by viewing the watershed as a lumped linear system.

7.1 GENERAL HYDROLOGIC SYSTEM MODEL

The amount of water stored in a hydrologic system, S may be related to the rates of inflow I and outflow Q by the integral equation of continuity (2.2.4):

$$\frac{dS}{dt} = I - Q \tag{7.1.1}$$

Imagine that the water is stored in a hydrologic system, such as a reservoir (Fig. 7.1.1), in which the amount of storage rises and falls with time in response to I and Q and their rates of change with respect to time: dI/dt, $d^2I/dt^2, \ldots,$ $dQ/dt, d^2Q/dt^2, \ldots$. Thus, the amount of storage at any time can be expressed by a *storage function* as:

$$S = f\left(I, \frac{dI}{dt}, \frac{d^2I}{dt^2}, \ldots, Q, \frac{dQ}{dt}, \frac{d^2Q}{dt^2}, \ldots\right) \tag{7.1.2}$$

The function f is determined by the nature of the hydrologic system being examined. For example, the linear reservoir introduced in Chap. 5 as a model for baseflow in streams relates storage and outflow by $S = kQ$, where k is a constant.

The continuity equation (7.1.1) and the storage function equation (7.1.2) must be solved simultaneously so that the output Q can be calculated given the input I, where I and Q are both functions of time. This can be done in two ways: by differentiating the storage function and substituting the result for dS/dt in (7.1.1), then solving the resulting differential equation in I and Q by integration; or by applying the finite difference method directly to Eqs. (7.1.1) and (7.1.2) to solve them recursively at discrete points in time. In this chapter, the first, or integral, approach is taken, and in Chap. 8, the second, or differential, approach is adopted.

Linear System in Continuous Time

For the storage function to describe a *linear system*, it must be expressed as a linear equation with constant coefficients. Equation (7.1.2) can be written

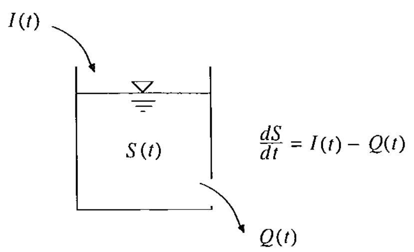


FIGURE 7.1.1
Continuity of water stored in a hydrologic system.

$$\begin{aligned} S = {} & a_1Q + a_2\frac{dQ}{dt} + a_3\frac{d^2Q}{dt^2} + \ldots + a_n\frac{d^{n-1}Q}{dt^{n-1}} \\ & + b_1I + b_2\frac{dI}{dt} + b_3\frac{d^2I}{dt^2} + \ldots + b_m\frac{d^{m-1}I}{dt^{m-1}} \end{aligned} \tag{7.1.3}$$

in which $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m$ are constants and derivatives of higher order than those shown are neglected. The constant coefficients also make the system *time-invariant* so that the way the system processes input into output does not change with time.

Differentiating (7.1.3), substituting the result for dS/dt in (7.1.1), and rearranging yields

$$\begin{aligned} & a_n\frac{d^nQ}{dt^n} + a_{n-1}\frac{d^{n-1}Q}{dt^{n-1}} + \ldots + a_2\frac{d^2Q}{dt^2} + a_1\frac{dQ}{dt} + Q = \\ & I - b_1\frac{dI}{dt} - b_2\frac{d^2I}{dt^2} - \ldots - b_{m-1}\frac{d^{m-1}I}{dt^{m-1}} - b_m\frac{d^mI}{dt^m} \end{aligned} \tag{7.1.4}$$

which may be rewritten in the more compact form

$$N(D)Q = M(D)I \tag{7.1.5}$$

where $D = d/dt$ and $N(D)$ and $M(D)$ are the differential operators

$$N(D) = a_n\frac{d^n}{dt^n} + a_{n-1}\frac{d^{n-1}}{dt^{n-1}} + \ldots + a_1\frac{d}{dt} + 1$$

and

$$M(D) = -b_m\frac{d^m}{dt^m} - b_{m-1}\frac{d^{m-1}}{dt^{m-1}} - \ldots - b_1\frac{d}{dt} + 1$$

Solving (7.1.5) for Q yields

$$Q(t) = \frac{M(D)}{N(D)}I(t) \tag{7.1.6}$$

The function $M(D)/N(D)$ is called the *transfer function* of the system; it describes the response of the output to a given input sequence.

Equation (7.1.4) was presented by Chow and Kulandaiswamy (1971) as a general hydrologic system model. It describes a lumped system because it contains derivatives with respect to time alone and not spatial dimensions. Chow and Kulandaiswamy showed that many of the previously proposed models of lumped hydrologic systems were special cases of this general model. For example, for a linear reservoir, the storage function (7.1.3) has $a_1 = k$ and all other coefficients zero, so (7.1.4) becomes

$$k\frac{dQ}{dt} + Q = I \tag{7.1.7}$$

7.2 RESPONSE FUNCTIONS OF LINEAR SYSTEMS

The solution of (7.1.6) for the transfer function of hydrologic systems follows two basic principles for linear system operations which are derived from methods for solving linear differential equations with constant coefficients (Kreyszig, 1968):

1. If a solution $f(Q)$ is multiplied by a constant c, the resulting function $cf(Q)$ is also a solution (*principle of proportionality*).
2. If two solutions $f_1(Q)$ and $f_2(Q)$ of the equation are added, the resulting function $f_1(Q) + f_2(Q)$ is also a solution of the equation (*principle of additivity* or *superposition*).

The particular solution adopted depends on the *input function* $N(D)I$, and on the specified *initial conditions* or values of the output variables at $t = 0$.

Impulse Response Function

The response of a linear system is uniquely characterized by its *impulse response function*. If a system receives an input of unit amount applied instantaneously (a unit impulse) at time τ, the response of the system at a later time t is described by the unit impulse response function $u(t - \tau)$; $t - \tau$ is the time lag since the impulse was applied [Fig. 7.2.1(*a*)]. The response of a guitar string when it is plucked is one example of a response to an impulse; another is the response of the shock absorber in a car after the wheel passes over a pothole. If the storage reservoir in Fig. 7.1.1 is initially empty, and then the reservoir is instantaneously filled with a unit amount of water, the resulting outflow function $Q(t)$ is the impulse response function.

Following the two principles of linear system operation cited above, if two impulses are applied, one of 3 units at time τ_1 and the other of 2 units at time τ_2, the response of the system will be $3u(t - \tau_1) + 2u(t - \tau_2)$, as shown in Fig. 7.2.1(*b*). Analogously, continuous input can be treated as a sum of infinitesimal impulses. The amount of input entering the system between times τ and $\tau + d\tau$ is $I(\tau)\,d\tau$. For example, if $I(\tau)$ is the precipitation intensity in inches per hour and $d\tau$ is an infinitesimal time interval measured in hours, then $I(\tau)\,d\tau$ is the depth in inches of precipitation input to the system during this interval. The direct runoff $t - \tau$ time units later resulting from this input is $I(\tau)u(t - \tau)d\tau$. The response to the complete input time function $I(\tau)$ can then be found by integrating the response to its constituent impulses:

$$Q(t) = \int_0^t I(\tau)u(t - \tau)\,d\tau \tag{7.2.1}$$

This expression, called the *convolution integral*, is the fundamental equation for solution of a linear system on a continuous time scale. Figure 7.2.2 illustrates the response summation process for the convolution integral.

For most hydrologic applications, solutions are needed at discrete intervals of time, because the input is specified as a discrete time function, such as an

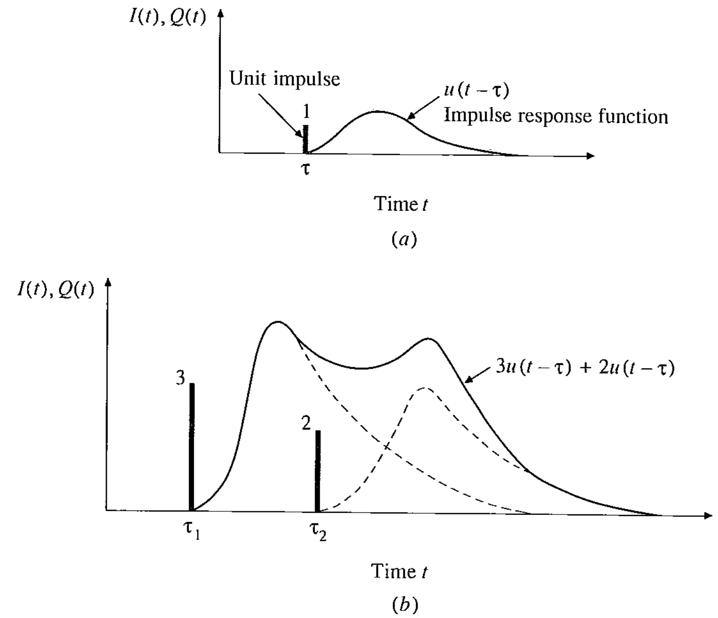


FIGURE 7.2.1
Responses of a linear system to impulse inputs. (*a*) Unit impulse response function. (*b*) The response to two impulses is found by summing the individual response functions.

excess rainfall hyetograph. To handle such input, two further functions are needed, the unit step response function and the unit pulse response function, as shown in Fig. 7.2.3.

Step Response Function

A *unit step input* is an input that goes from a rate of 0 to 1 at time 0 and continues indefinitely at that rate thereafter [Fig. 7.2.3(*b*)]. The output of the system, its *unit step response function* $g(t)$ is found from (7.2.1) with $I(\tau) = 1$ for $\tau \geq 0$, as

$$Q(t) = g(t) = \int_0^t u(t - \tau)\,d\tau \tag{7.2.2}$$

If the substitution $l = t - \tau$ is made in (7.2.2) then $d\tau = -dl$, the limit $\tau = t$ becomes $l = t - t = 0$, and the limit $\tau = 0$ becomes $l = t - 0 = t$. Hence,

$$g(t) = -\int_t^0 u(l)\,dl$$

or

$$g(t) = \int_0^t u(l)\,dl \tag{7.2.3}$$

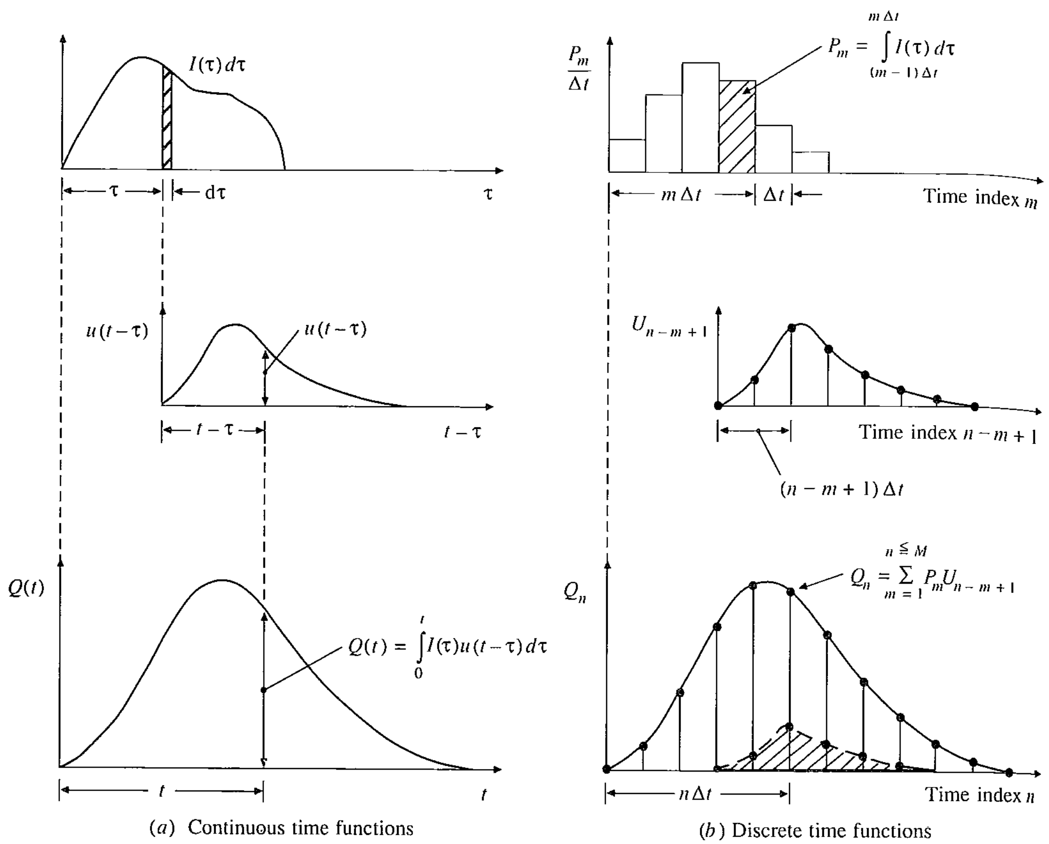


FIGURE 7.2.2
The relationship between continuous and discrete convolution.

In words, the value of the unit step response function $g(t)$ at time t equals the integral of the impulse response function up to that time, as shown in Fig. 7.2.3(a) and (b).

Pulse Response Function

A *unit pulse input* is an input of unit amount occurring in duration Δt. The rate is $I(\tau) = 1/\Delta t$, $0 \le \tau \le \Delta t$, and zero elsewhere. The *unit pulse response function* produced by this input can be found by the two linear system principles cited earlier. First, by the principle of proportionality, the response to a unit step input of rate $1/\Delta t$ beginning at time 0 is $(1/\Delta t)g(t)$. If a similar unit step input began at time Δt instead of at 0, its response function would be lagged by time interval Δt, and would have a value at time t equal to $(1/\Delta t)g(t - \Delta t)$. Then, using the principle of superposition, the response to a unit pulse input duration Δt is found by subtracting the response to a step input of rate $1/\Delta t$ beginning at time Δt from

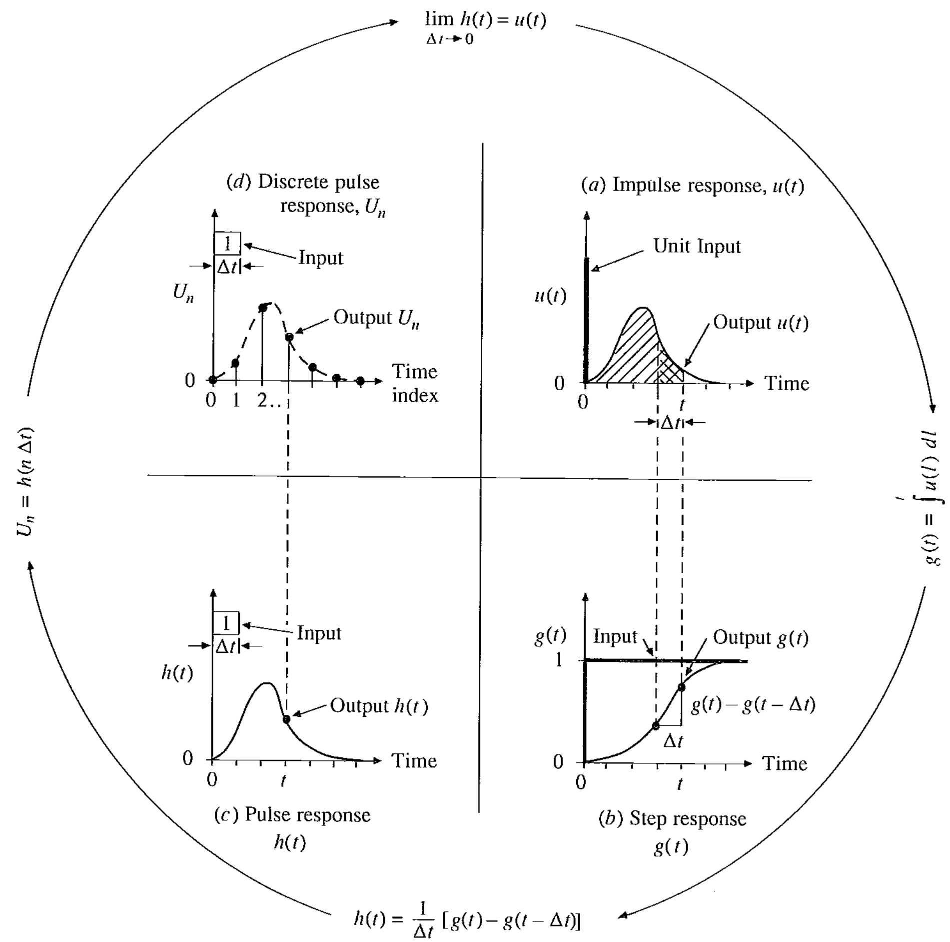


FIGURE 7.2.3
Response functions of a linear system. The response functions in (a), (b), and (c) are on a continuous time domain and that in (d) on a discrete time domain.

the response to a step input of the same rate beginning at time 0, so that the *unit pulse response function* $h(t)$ is

$$h(t) = \frac{1}{\Delta t}[g(t) - g(t - \Delta t)] \tag{7.2.4}$$

$$= \frac{1}{\Delta t}\left[\int_0^t u(l)\,dl - \int_0^{t-\Delta t} u(l)\,dl\right]$$

$$= \frac{1}{\Delta t}\int_{t-\Delta t}^{t} u(l)\,dl \tag{7.2.5}$$

As shown in Fig. 7.2.3, $g(t)-g(t-\Delta t)$ represents the area under the impulse response function between $t-\Delta t$ and t, and $h(t)$ represents the slope of the unit step response function $g(t)$ between these two time points.

Example 7.2.1. Determine the impulse, step and pulse response functions of a linear reservoir with storage constant $k\,(S = kQ)$.

Solution. The continuity equation (7.1.1) is

$$\frac{dS}{dt} = I(t) - Q(t)$$

and differentiating the storage function $S = kQ$ yields $dS/dt = k\,dQ/dt$, so

$$k\frac{dQ}{dt} = I(t) - Q(t)$$

or

$$\frac{dQ}{dt} + \frac{1}{k}Q(t) = \frac{1}{k}I(t)$$

This is a first-order linear differential equation, and can be solved by multiplying both sides of the equation by the *integrating factor* $e^{t/k}$:

$$e^{t/k}\frac{dQ}{dt} + \frac{1}{k}e^{t/k}Q(t) = \frac{1}{k}e^{t/k}I(t)$$

so that the two terms on the left-hand side of the equation can be combined as

$$\frac{d}{dt}(Qe^{t/k}) = \frac{1}{k}e^{t/k}I(t)$$

Integrating from the initial conditions $Q = Q_o$ at $t = 0$

$$\int_{Q_o,0}^{Q(t),t} d(Qe^{t/k}) = \int_0^t \frac{1}{k}e^{\tau/k}I(\tau)\,d\tau$$

where τ is a dummy variable of time in the integration. Solving,

$$Q(t)e^{t/k} - Q_o = \int_0^t \frac{1}{k}e^{\tau/k}I(\tau)\,d\tau$$

and rearranging,

$$Q(t) = Q_o e^{-t/k} + \int_0^t \frac{1}{k}e^{-(t-\tau)/k}I(\tau)\,d\tau$$

Comparing this equation with the convolution integral (7.2.1), it can be seen that the two equations are the same provided $Q_o = 0$ and

$$u(t-\tau) = \frac{1}{k}e^{-(t-\tau)/k}$$

So if l is defined as the lag time $t-\tau$, the impulse response function of a linear reservoir is

$$u(l) = \frac{1}{k}e^{-l/k}$$

The requirement that $Q_o = 0$ implies that the system starts from rest when the convolution integral is applied.

The unit step response is given by (7.2.3):

$$\begin{aligned} g(t) &= \int_o^t u(l)\,dl \\ &= \int_o^t \frac{1}{k}e^{-l/k}dl \\ &= [-e^{-l/k}]_o^t \\ &= 1 - e^{-t/k} \end{aligned}$$

The unit pulse response is given by (7.2.4):

$$h(t) = \frac{1}{\Delta t}[g(t) - g(t-\Delta t)]$$

1. For $0 \le t \le \Delta t$, $g(t-\Delta t) = 0$, so

$$h(t) = \frac{1}{\Delta t}g(t) = \frac{1}{\Delta t}(1 - e^{-t/k})$$

2. For $t > \Delta t$,

$$\begin{aligned} h(t) &= \frac{1}{\Delta t}[1 - e^{-t/k} - (1 - e^{-(t-\Delta t)/k})] \\ &= \frac{e^{-t/k}}{\Delta t}(e^{\Delta t/k} - 1) \end{aligned}$$

The impulse and step response functions of a linear reservoir with $k = 3$ h are plotted in Fig. 7.2.4, along with the pulse response function for $\Delta t = 2$ h.

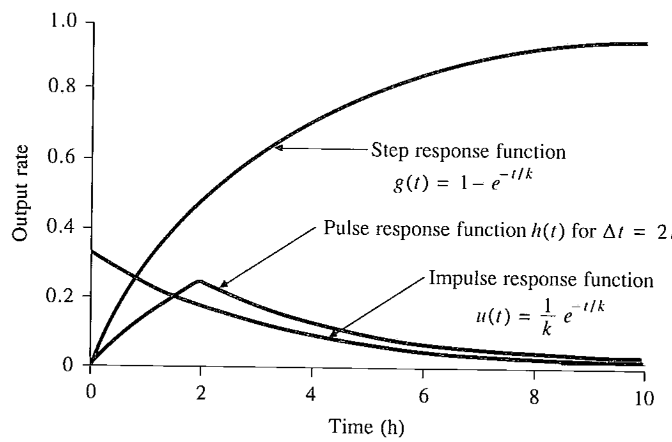


FIGURE 7.2.4
Response function of a linear reservoir with $k = 3$ h. Pulse response function is for a pulse input of two hours duration. (from Example 7.2.1.)

Linear System in Discrete Time

The impulse, step, and pulse response functions have all been defined on a continuous time domain. Now let the time domain be broken into discrete intervals of duration Δt. As shown in Sec. 2.3, there are two ways to represent a continuous time function on a discrete time domain, as a *pulse data system* or as a *sample data system*. The pulse data system is used for precipitation and the value of its discrete input function for the mth time interval is

$$P_m = \int_{(m-1)\Delta t}^{m\Delta t} I(\tau)\,dt \qquad m = 1, 2, 3, \ldots \tag{7.2.6}$$

P_m is the depth of precipitation falling during the time interval (in inches or centimeters). The sample data system is used for streamflow and direct runoff, so that the value of the system output in the nth time interval ($t = n\,\Delta t$) is

$$Q_n = Q(n\Delta t) \qquad n = 1, 2, 3, \ldots \tag{7.2.7}$$

Q_n is the instantaneous value of the flow rate at the end of the nth time interval (in cfs or m^3/s). Thus the input and output variables to a watershed system are recorded with different dimensions and using different discrete data representations. The effect of an input pulse of duration Δt beginning at time $(m-1)\Delta t$ on the output at time $t = n\Delta t$ is measured by the value of the unit pulse response function $h[t - (m-1)\Delta t] = h[n\Delta t - (m-1)\Delta t] = h[(n - m + 1)\Delta t]$, given, following Eq. (7.2.5), as

$$h[(n - m + 1)\Delta t] = \frac{1}{\Delta t}\int_{(n-m)\Delta t}^{(n-m+1)\Delta t} u(l)\,dl \tag{7.2.8}$$

On a discrete time domain, the input function is a series of M pulses of constant rate: for pulse m, $I(\tau) = P_m/\Delta t$ for $(m-1)\,\Delta t \le \tau \le m\,\Delta t$. $I(\tau) = 0$ for $\tau > M\Delta t$. Consider the case where the output is being calculated after all the input has ceased, that is, at $t = n\Delta t > M\Delta t$ [see Fig. 7.2.2(*b*)]. The contribution to the output of each of the M input pulses can be found by breaking the convolution integral (7.2.1) at $t = n\Delta t$ into M parts:

$$\begin{aligned} Q_n &= \int_0^{n\Delta t} I(\tau)u(n\Delta t - \tau)d\tau \\ &= \frac{P_1}{\Delta t}\int_0^{\Delta t} u(n\Delta t - \tau)\,d\tau + \frac{P_2}{\Delta t}\int_{\Delta t}^{2\Delta t} u(n\Delta t - \tau)\,d\tau + \ldots \\ &\quad + \frac{P_m}{\Delta t}\int_{(m-1)\Delta t}^{m\Delta t} u(n\Delta t - \tau)\,d\tau + \ldots + \frac{P_M}{\Delta t}\int_{(M-1)\Delta t}^{M\Delta t} u(n\Delta t - \tau)\,d\tau \end{aligned} \tag{7.2.9}$$

where the terms $P_m/\Delta t$, $m = 1, 2, \ldots, M$, can be brought outside the integrals because they are constants.

In each of these integrals, the substitution $l = n\Delta t - \tau$ is made, so $d\tau = -dl$, the limit $\tau = (m-1)\,\Delta t$ becomes $l = n\Delta t - (m-1)\,\Delta t = (n - m + 1)\Delta t$, and the limit $\tau = m\Delta t$ becomes $l = (n-m)\Delta t$. The mth integral in (7.2.9) is now written

$$\begin{aligned} \frac{P_m}{\Delta t}\int_{(m-1)\Delta t}^{m\Delta t} u(n\Delta t - \tau)\,d\tau &= \frac{P_m}{\Delta t}\int_{(n-m+1)\Delta t}^{(n-m)\Delta t} -u(l)\,dl \\ &= \frac{P_m}{\Delta t}\int_{(n-m)\Delta t}^{(n-m+1)\Delta t} u(l)\,dl \\ &= P_m h[(n - m + 1)\Delta t] \end{aligned} \tag{7.2.10}$$

by substitution from (7.2.7). After making these substitutions for each term in (7.2.9),

$$\begin{aligned} Q_n &= P_1 h[(n\Delta t)] + P_2 h[(n-1)\Delta t] + \ldots \\ &\quad + P_m h[(n - m + 1)\Delta t] + \ldots \\ &\quad + P_M h[(n - M + 1)\Delta t] \end{aligned} \tag{7.2.11}$$

which is a convolution equation with input P_m in pulses and output Q_n as a sample data function of time.

Discrete Pulse Response Function

As shown in Fig. 7.2.3(*d*), the continuous pulse response function $h(t)$ may be represented on a discrete time domain as a sample data function U where

$$U_{n-m+1} = h[(n - m + 1)\Delta t] \tag{7.2.12}$$

It follows that $U_n = h[n\Delta t]$, $U_{n-1} = h[(n-1)\Delta t], \ldots,$ and $U_{n-M+1} = h[(n - M + 1)\Delta t]$. Substituting into (7.2.11), the discrete-time version of the convolution integral is

$$\begin{aligned} Q_n &= P_1 U_n + P_2 U_{n-1} + \ldots + P_m U_{n-m+1} + \ldots + P_M U_{n-M+1} \\ &= \sum_{m=1}^{M} P_m U_{n-m+1} \end{aligned} \tag{7.2.13}$$

Equation (7.2.13) is valid provided $n \ge M$; if $n < M$, then, in (7.2.9), one would only need to account for the first n pulses of input, since these are the only pulses that can influence the output up to time $n\,\Delta t$. In this case, (7.2.13) is rewritten

$$Q_n = \sum_{m=1}^{n} P_m U_{n-m+1} \tag{7.2.14}$$

Combining (7.2.13) and (7.2.14) gives the final result

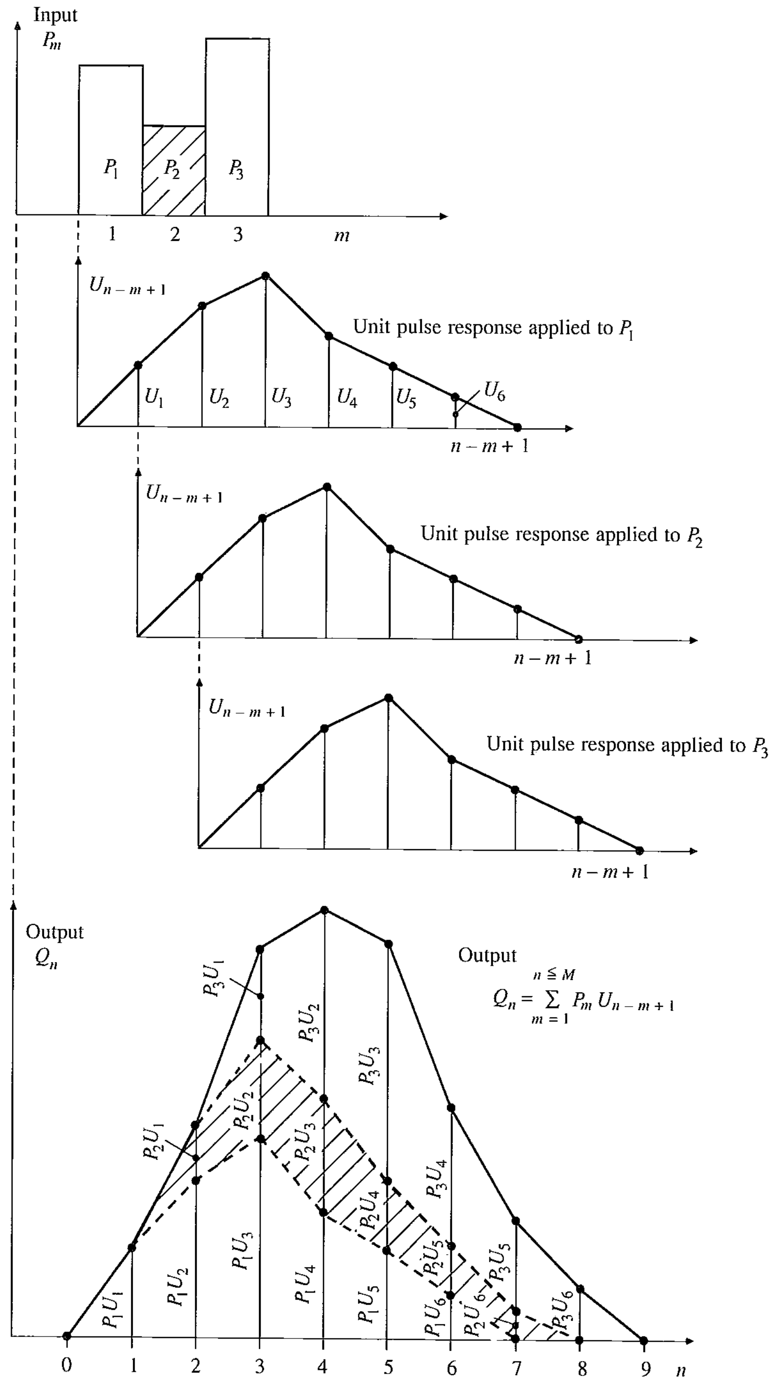


FIGURE 7.2.5
Application of the discrete convolution equation to the output from a linear system.

$$Q_n = \sum_{m=1}^{n \le M} P_m U_{n-m+1} \tag{7.2.15}$$

which is the *discrete convolution equation* for a linear system. The notation $n \le M$ as the upper limit of the summation shows that the terms are summed for $m = 1, 2, \ldots, n$ for $n \le M$, but for $n > M$, the summation is limited to $m = 1, 2, \ldots, M$.

As an example, suppose there are $M = 3$ pulses of input: P_1, P_2, and P_3. For the first time interval ($n = 1$), there is only one term in the convolution, that for $m = 1$;

$$Q_1 = P_1U_{1-1+1} = P_1U_1$$

For $n = 2$, there are two terms, corresponding to $m = 1, 2$:

$$Q_2 = P_1U_{2-1+1} + P_2U_{2-2+1} = P_1U_2 + P_2U_1$$

For $n = 3$, there are three terms:

$$Q_3 = P_1U_{3-1+1} + P_2U_{3-2+1} + P_3U_{3-3+1} = P_1U_3 + P_2U_2 + P_3U_1$$

And for $n = 4, 5, \ldots$ there continue to be just three terms:

$$Q_n = P_1U_n + P_2U_{n-1} + P_3U_{n-2}$$

The results of the calculation are shown diagramatically in Fig. 7.2.5. The sum of the subscripts in each term on the right-hand side of the summation is always one greater than the subscript of Q.

In the example shown in the diagram, there are 3 input pulses and 6 non-zero terms in the pulse response function U, so there are $3 + 6 - 1 = 8$ non-zero terms in the output function Q. The values of the output for the final three periods are:

$$Q_6 = P_1U_6 + P_2U_5 + P_3U_4$$

$$Q_7 = P_2U_6 + P_3U_5$$

$$Q_8 = P_3U_6$$

Q_n and P_m are expressed in different dimensions, and U has dimensions that are the ratio of the dimensions of Q_n and P_m to make (7.2.15) dimensionally consistent. For example, if P_m is measured in inches and Q_n in cfs, then the dimensions of U are cfs/in, which may be interpreted as cfs of output per inch of input.

7.3 THE UNIT HYDROGRAPH

The unit hydrograph is the unit pulse response function of a linear hydrologic system. First proposed by Sherman (1932), the unit hydrograph (originally named *unit-graph*) of a watershed is defined as a direct runoff hydrograph (DRH) resulting from 1 in (usually taken as 1 cm in SI units) of excess rainfall generated uniformly over the drainage area at a constant rate for an effective duration. Sherman originally used the word "unit" to denote a unit of time, but since that time it has often been interpreted as a unit depth of excess rainfall. Sherman classified runoff into surface runoff and groundwater runoff and defined the unit hydrograph for use only with surface runoff. Methods of calculating excess rainfall and direct runoff from observed rainfall and streamflow data are presented in Chap. 5.

The unit hydrograph is a simple linear model that can be used to derive the hydrograph resulting from any amount of excess rainfall. The following basic assumptions are inherent in this model:

1. The excess rainfall has a constant intensity within the effective duration.
2. The excess rainfall is uniformly distributed throughout the whole drainage area.
3. The base time of the DRH (the duration of direct runoff) resulting from an excess rainfall of given duration is constant.
4. The ordinates of all DRH's of a common base time are directly proportional to the total amount of direct runoff represented by each hydrograph.
5. For a given watershed, the hydrograph resulting from a given excess rainfall reflects the unchanging characteristics of the watershed.

Under natural conditions, the above assumptions cannot be perfectly satisfied. However, when the hydrologic data to be used are carefully selected so that they come close to meeting the above assumptions, the results obtained by the unit hydrograph model are generally acceptable for practical purposes (Heerdegen, 1974). Although the model was originally devised for large watersheds, it has been found applicable to small watersheds from less than 0.5 hectares to 25 km^2 (about 1 acre to 10 mi^2). Some cases do not support the use of the model because one or more of the assumptions are not well satisfied. For such reasons, the model is considered inapplicable to runoff originating from snow or ice.

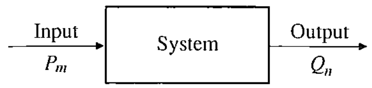
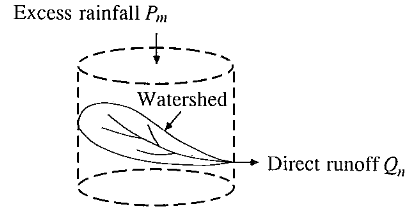
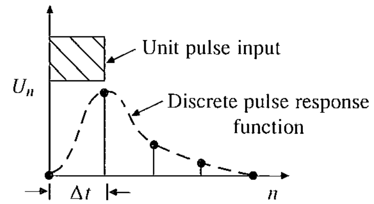
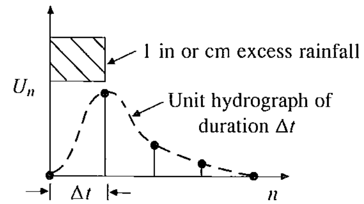
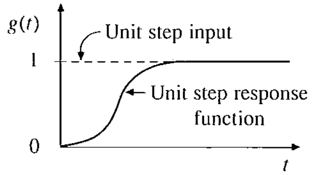
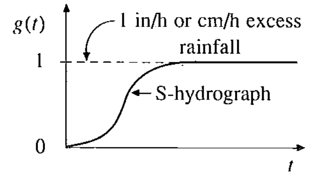
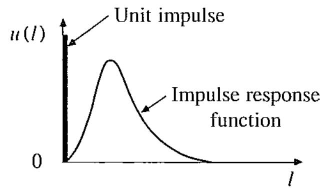
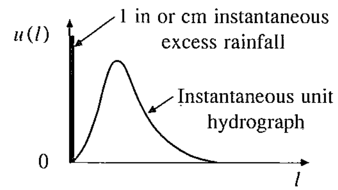
Concerning assumption (1), the storms selected for analysis should be of short duration, since these will most likely produce an intense and nearly constant excess rainfall rate, yielding a well-defined single-peaked hydrograph of short time base.

Concerning assumption (2), the unit hydrograph may become inapplicable when the drainage area is too large to be covered by a nearly uniform distribution of rainfall. In such cases, the area has to be divided and each subarea analyzed for storms covering the whole subarea.

Concerning assumption (3), the base time of the direct runoff hydrograph (DRH) is generally uncertain but depends on the method of baseflow separation (see Sec. 5.2). The base time is usually short if the direct runoff is considered to include the surface runoff only; it is long if the direct runoff also includes subsurface runoff.

Concerning assumption (4), the principles of superposition and proportionality are assumed so that the ordinates Q_n of the DRH may be computed by Eq.

TABLE 7.3.1
Comparison of linear system and unit hydrograph concepts

Linear system	Unit hydrograph
1.  $Q_n = \sum_{m=1}^{n \le M} P_m U_{n-m+1}$	1. 
2. 	2. 
3. 	3. 
4. 	4. 
5. System starts from rest.	5. Direct runoff hydrograph starts from zero. All previous rainfall is absorbed by watershed (initial abstraction or loss).
6. System is linear.	6. Direct runoff hydrograph is calculated using principles of proportionality and superposition.
7. Transfer function has constant coefficients.	7. Watershed response is time invariant, not changing from one storm to another.
8. System obeys continuity. $\frac{dS}{dt} = I(t) - Q(t)$	8. Total depths of excess rainfall and direct runoff are equal. $\sum_n Q_n = \sum_m P_m$

(7.2.15). Actual hydrologic data are not truly linear; when applying (7.2.15) to them, the resulting hydrograph is only an approximation, which is satisfactory in many practical cases.

Concerning assumption (5), the unit hydrograph is considered unique for a given watershed and invariable with respect to time. This is the *principle of time invariance*, which, together with the principles of superposition and proportionality, is fundamental to the unit hydrograph model. Unit hydrographs are applicable only when channel conditions remain unchanged and watersheds do not have appreciable storage. This condition is violated when the drainage area contains many reservoirs, or when the flood overflows into the flood plain, thereby producing considerable storage.

The principles of linear system analysis form the basis of the unit hydrograph method. Table 7.3.1 shows a comparison of linear system concepts with the corresponding unit hydrograph concepts. In hydrology, the step response function is commonly called the *S-hydrograph*, and the impulse response function is called the *instantaneous unit hydrograph* which is the hypothetical response to a unit depth of excess rainfall deposited instantaneously on the watershed surface.

7.4 UNIT HYDROGRAPH DERIVATION

The discrete convolution equation (7.2.15) allows the computation of direct runoff Q_n given excess rainfall P_m and the unit hydrograph U_{n-m+1}

$$Q_n = \sum_{m=1}^{n\leq M} P_m U_{n-m+1} \tag{7.4.1}$$

The reverse process, called *deconvolution*, is needed to derive a unit hydrograph given data on P_m and Q_n. Suppose that there are M pulses of excess rainfall and N pulses of direct runoff in the storm considered; then N equations can be written for $Q_n, n = 1, 2, \ldots, N$, in terms of $N - M + 1$ unknown values of the unit hydrograph, as shown in Table 7.4.1.

If Q_n and P_m are given and U_{n-m+1} is required, the set of equations in Table 7.4.1 is *overdetermined*, because there are more equations (N) than unknowns ($N - M + 1$).

Example 7.4.1. Find the half-hour unit hydrograph using the excess rainfall hyetograph and direct runoff hydrograph given in Table 7.4.2. (these were derived in Example 5.3.1.)

Solution. The ERH and DRH in Table 7.4.2 have $M = 3$ and $N = 11$ pulses respectively. Hence, the number of pulses in the unit hydrograph is $N - M + 1 = 11 - 3 + 1 = 9$. Substituting the ordinates of the ERH and DRH into the equations in Table 7.4.1 yields a set of 11 simultaneous equations. These equations may be solved by *Gauss elimination* to give the unit hydrograph ordinates. Gauss elimination involves isolating the unknown variables one by one and successively solving for them. In this case, the equations can be solved from top to bottom, working with just the equations involving the first pulse P_1, starting with

TABLE 7.4.1

The set of equations for discrete time convolution $Q_n = \sum_{m=1}^{n\leq M} P_m U_{n-m+1}$; $n = 1, 2, \ldots, N$

$$\begin{aligned}
Q_1 &= P_1U_1 \\
Q_2 &= P_2U_1 + P_1U_2 \\
Q_3 &= P_3U_1 + P_2U_2 + P_1U_3 \\
&\ldots \\
Q_M &= P_MU_1 + P_{M-1}U_2 + \cdots + P_1U_M \\
Q_{M+1} &= 0 + P_MU_2 + \ldots + P_2U_M + P_1U_{M+1} \\
&\ldots \\
Q_{N-1} &= 0 + 0 + \ldots + 0 + 0 + \ldots + P_MU_{N-M} + P_{M-1}U_{N-M+1} \\
Q_N &= 0 + 0 + \ldots + 0 + 0 + \ldots + 0 + P_MU_{N-M+1}
\end{aligned}$$

$$U_1 = \frac{Q_1}{P_1} = \frac{428}{1.06} = 404 \text{ cfs/in}$$

$$U_2 = \frac{Q_2 - P_2U_1}{P_1} = \frac{1923 - 1.93 \times 404}{1.06} = 1079 \text{ cfs/in}$$

$$U_3 = \frac{Q_3 - P_3U_1 - P_2U_2}{P_1} = \frac{5297 - 1.81 \times 404 - 1.93 \times 1079}{1.06} = 2343 \text{ cfs/in}$$

and similarly for the remaining ordinates

$$U_4 = \frac{9131 - 1.81 \times 1079 - 1.93 \times 2343}{1.06} = 2506 \text{ cfs/in}$$

$$U_5 = \frac{10625 - 1.81 \times 2343 - 1.93 \times 2506}{1.06} = 1460 \text{ cfs/in}$$

$$U_6 = \frac{7834 - 1.81 \times 2506 - 1.93 \times 1460}{1.06} = 453 \text{ cfs/in}$$

TABLE 7.4.2
Excess rainfall hyetograph and direct runoff hydrograph for Example 7.4.1

Time ($\frac{1}{2}$ h)	Excess rainfall (in)	Direct runoff (cfs)
1	1.06	428
2	1.93	1923
3	1.81	5297
4		9131
5		10625
6		7834
7		3921
8		1846
9		1402
10		830
11		313

TABLE 7.4.3
Unit hydrograph derived in Example 7.4.1

n	1	2	3	4	5	6	7	8	9
U_n (cfs/in)	404	1079	2343	2506	1460	453	381	274	173

$$U_7 = \frac{3921 - 1.81 \times 1460 - 1.93 \times 453}{1.06} = 381 \text{ cfs/in}$$

$$U_8 = \frac{1846 - 1.81 \times 453 - 1.93 \times 381}{1.06} = 274 \text{ cfs/in}$$

$$U_9 = \frac{1402 - 1.81 \times 381 - 1.93 \times 274}{1.06} = 173 \text{ cfs/in}$$

The derived unit hydrograph is given in Table 7.4.3. Solutions may be similarly obtained by focusing on other rainfall pulses. The depth of direct runoff in the unit hydrograph can be checked and found to equal 1.00 inch as required. In cases where the derived unit hydrograph does not meet this requirement, the ordinates are adjusted by proportion so that the depth of direct runoff is 1 inch (or 1 cm).

In general the unit hydrographs obtained by solutions of the set of equations in Table 7.4.1 for different rainfall pulses are not identical. To obtain a unique solution a *method of successive approximation* (Collins, 1939) can be used, which involves four steps: (1) assume a unit hydrograph, and apply it to all excess-rainfall blocks of the hyetograph except the largest; (2) subtract the resulting hydrograph from the actual DRH, and reduce the residual to unit hydrograph terms; (3) compute a weighted average of the assumed unit hydrograph and the residual unit hydrograph, and use it as the revised approximation for the next trial; (4) repeat the previous three steps until the residual unit hydrograph does not differ by more than a permissible amount from the assumed hydrograph.

The resulting unit hydrograph may show erratic variations and even have negative values. If this occurs, a smooth curve may be fitted to the ordinates to produce an approximation of the unit hydrograph. Erratic variation in the unit hydrograph may be due to nonlinearity in the effective rainfall–direct runoff relationship in the watershed, and even if this relationship is truly linear, the observed data may not adequately reflect this. Also, actual storms are not always uniform in time and space, as required by theory, even when the excess rainfall hyetograph is broken into pulses of short duration.

7.5 UNIT HYDROGRAPH APPLICATION

Once the unit hydrograph has been determined, it may be applied to find the direct runoff and streamflow hydrographs. A rainfall hyetograph is selected, the abstractions are estimated, and the excess rainfall hyetograph is calculated as described in Sec. 5.4. The time interval used in defining the excess rainfall hyetograph ordinates must be the same as that for which the unit hydrograph was specified. The discrete convolution equation

$$Q_n = \sum_{m=1}^{n \le M} P_m U_{n-m+1} \qquad (7.5.1)$$

may then be used to yield the direct runoff hydrograph. By adding an estimated baseflow to the direct runoff hydrograph, the streamflow hydrograph is obtained.

Example 7.5.1. Calculate the streamflow hydrograph for a storm of 6 in excess rainfall, with 2 in in the first half-hour, 3 in in the second half-hour and 1 in in the third half-hour. Use the half-hour unit hydrograph computed in Example 7.4.1 and assume the baseflow is constant at 500 cfs throughout the flood. Check that the total depth of direct runoff is equal to the total excess precipitation (watershed area $= 7.03$ mi^2).

Solution. The calculation of the direct runoff hydrograph by convolution is shown in Table 7.5.1. The unit hydrograph ordinates from Table 7.4.3 are laid out along the top of the table and the excess precipitation depths down the left side. The time interval is in $\Delta t = 0.5$ h intervals. For the first time interval, $n = 1$ in Eq. (7.5.1), and

$$\begin{aligned} Q_1 &= P_1 U_1 \\ &= 2.00 \times 404 \\ &= 808 \text{ cfs} \end{aligned}$$

For the second time interval,

$$\begin{aligned} Q_2 &= P_2 U_1 + P_1 U_2 \\ &= 3.00 \times 404 + 2.00 \times 1079 \\ &= 1212 + 2158 \end{aligned}$$

TABLE 7.5.1
Calculation of the direct runoff hydrograph and streamflow hydrograph for Example 7.5.1

		Unit hydrograph ordinates (cfs/in)										
Time ($\frac{1}{2}$-h)	**Excess Precipitation (in)**	**1 404**	**2 1079**	**3 2343**	**4 2506**	**5 1460**	**6 453**	**7 381**	**8 274**	**9 173**	**Direct runoff (cfs)**	**Streamflow* (cfs)**
$n =$ 1	2.00	808									808	1308
2	3.00	1212	2158								3370	3870
3	1.00	404	3237	4686							8327	8827
4			1079	7029	5012						13,120	13,620
5				2343	7518	2920					12,781	13,281
6					2506	4380	906				7792	8292
7						1460	1359	762			3581	4081
8							453	1143	548		2144	2644
9								381	822	346	1549	2049
10									274	519	793	1293
11										173	173	673
										Total	54,438	

*Baseflow = 500 cfs.

$$= 3370 \text{ cfs}$$

as shown in the table. For the third time interval,

$$\begin{aligned} Q_3 &= P_3U_1 + P_2U_2 + P_1U_3 \\ &= 1.00 \times 404 + 3.00 \times 1079 + 2.00 \times 2343 \\ &= 404 + 3237 + 4686 \\ &= 8327 \text{ cfs} \end{aligned}$$

The calculations for $n = 4, 5, \ldots,$ follow in the same manner as shown in Table 7.5.1 and graphically in Fig. 7.5.1. The total direct runoff volume is

$$\begin{aligned} V_d &= \sum_{n=1}^{N} Q_n \Delta t \\ &= 54{,}438 \times 0.5 \text{ cfs}\cdot\text{h} \\ &= 54{,}438 \times 0.5 \frac{\text{ft}^3\cdot\text{h}}{\text{s}} \times \frac{3600 \text{ s}}{1 \text{ h}} \\ &= 9.80 \times 10^7 \text{ ft}^3 \end{aligned}$$

and the corresponding depth of direct runoff is found by dividing by the watershed area $A = 7.03 \text{ mi}^2 = 7.03 \times 5280^2 \text{ ft}^2 = 1.96 \times 10^8 \text{ ft}^2$:

$$\begin{aligned} r_d &= \frac{V_d}{A} \\ &= \frac{9.80 \times 10^7}{1.96 \times 10^8} \text{ft} \\ &= 0.500 \text{ ft} \\ &= 6.00 \text{ in} \end{aligned}$$

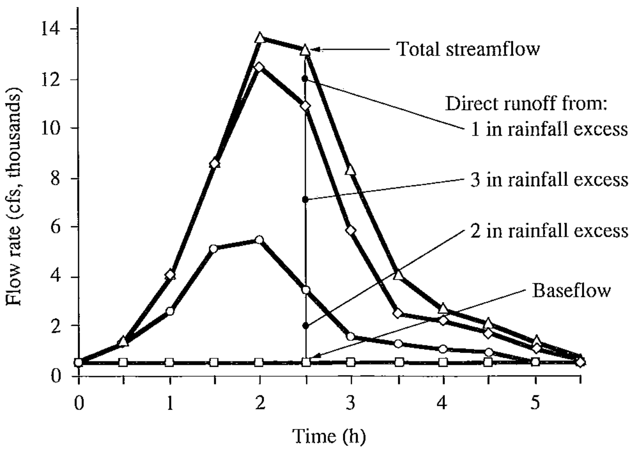


FIGURE 7.5.1
Streamflow hydrograph from a storm with excess rainfall pulses of duration 0.5 h and amount 2 in, 3 in, and 1 in, respectively. Total streamflow = baseflow + direct runoff (Example 7.5.1).

which is equal to the total depth of excess precipitation as required.

The streamflow hydrograph is found by adding the 500 cfs baseflow to the direct runoff hydrograph, as shown on the right-hand side of Table 7.5.1 and graphically in Fig. 7.5.1.

7.6 UNIT HYDROGRAPH BY MATRIX CALCULATION

Deconvolution may be used to derive the unit hydrograph from a complex multipeaked hydrograph, but the possibility of errors or nonlinearity in the data is greater than for a single-peaked hydrograph. Least-squares fitting or an optimization method can be used to minimize the error in the fitted direct runoff hydrograph. The application of these techniques is facilitated by expressing Eq. (7.4.1) in matrix form:

$$\begin{bmatrix} P_1 & 0 & 0 & \ldots 0 & 0 & \ldots 0 & 0 \\ P_2 & P_1 & 0 & \ldots 0 & 0 & \ldots 0 & 0 \\ P_3 & P_2 & P_1 & \ldots 0 & 0 & \ldots 0 & 0 \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ P_M & P_{M-1} & P_{M-2} & \ldots P_1 & 0 & \ldots 0 & 0 \\ 0 & P_M & P_{M-1} & \ldots P_2 & P_1 & \ldots 0 & 0 \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ 0 & 0 & 0 & \ldots 0 & 0 & \ldots P_M & P_{M-1} \\ 0 & 0 & 0 & \ldots 0 & 0 & \ldots 0 & P_M \end{bmatrix} \cdot \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ \vdots \\ U_{N-M+1} \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ \cdot \\ \cdot \\ \cdot \\ Q_M \\ Q_{M+1} \\ \cdot \\ \cdot \\ \cdot \\ Q_{N-1} \\ Q_N \end{bmatrix} \qquad (7.6.1)$$

or

$$[P][U] = [Q] \qquad (7.6.2)$$

Given $[P]$ and $[Q]$, there is usually no solution for $[U]$ that will satisfy all N equations (7.6.1). Suppose that a solution $[U]$ is given that yields an estimate $[\hat{Q}]$ of the DRH as

$$[P][U] = [\hat{Q}] \qquad (7.6.3a)$$

or

$$\hat{Q}_n = P_nU_1 + P_{n-1}U_2 + \ldots + P_{n-M+1}U_M \qquad n = 1, \ldots, N \qquad (7.6.3b)$$

with all equations now satisfied. A solution is sought which minimizes the error $[Q] - [\hat{Q}]$ between the observed and estimated DRH's.

Solution by Linear Regression

The solution by linear regression produces the least-squares error between $[Q]$ and $[\hat{Q}]$ (Snyder, 1955). To solve Eq. (7.6.2) for $[U]$, the rectangular matrix $[P]$

is reduced to a square matrix $[Z]$ by multiplying both sides by the transpose of $[P]$, denoted by $[P]^T$, which is formed by interchanging the rows and columns of $[P]$. Then both sides are multiplied by the inverse $[Z]^{-1}$ of matrix $[Z]$, to yield

$$[U] = [Z]^{-1}[P]^T[Q] \tag{7.6.4}$$

where $[Z] = [P]^T[P]$. However, the solution is not easy to determine by this method, because the many repeated and blank entries in $[P]$ create difficulties in the inversion of $[Z]$ (Bree, 1978). Newton and Vinyard (1967) and Singh (1976) give alternative methods of obtaining the least-squares solution, but these methods do not ensure that all the unit hydrograph ordinates will be nonnegative.

Solution by Linear Programming

Linear programming is an alternative method of solving for $[U]$ in Eq. (7.6.2) that minimizes the absolute value of the error between $[Q]$ and $[\hat{Q}]$ and also ensures that all entries of $[U]$ are nonnegative (Eagleson, Mejia, and March, 1966; Deininger, 1969; Singh, 1976; Mays and Coles, 1980).

The general linear programming model is stated in the form of a linear *objective function* to be optimized (maximized or minimized) subject to linear *constraint equations*. Linear programming provides a method of comparing all possible solutions that satisfy the constraints and obtaining the one that optimizes the objective function (Hillier and Lieberman, 1974; Bradley, Hax, and Magnanti, 1977).

Example 7.6.1. Develop a linear program to solve Eq. (7.6.2) for the unit hydrograph given the ERH $P_m, m = 1, 2, \ldots, M$, and the DRH $Q_n, n = 1, 2, \ldots, N$.

Solution. The objective is to minimize $\sum_{n=1}^{N} |\epsilon_n|$ where $\epsilon_n = Q_n - \hat{Q}_n$. Linear programming requires that all the variables be nonnegative; to accomplish this task, ϵ_n is split into two components, a *positive deviation* θ_n and a *negative deviation* β_n. In the case where $\epsilon_n > 0$, that is, when the observed direct runoff Q_n is greater than the calculated value $\hat{Q}_n$, $\theta_n = \epsilon_n$ and $\beta_n = 0$; where $\epsilon_n < 0$, $\beta_n = -\epsilon_n$ and $\theta_n = 0$ (see Fig. 7.6.1). If $\epsilon_n = 0$ then $\theta_n = \beta_n = 0$ also. Hence, the solution must obey

$$Q_n = \hat{Q}_n - \beta_n + \theta_n \qquad n = 1, 2, \ldots, N \tag{7.6.5}$$

and the objective is

$$\text{minimize} \sum_{n=1}^{N} (\theta_n + \beta_n) \tag{7.6.6}$$

The constraints (7.6.5) can be written

$$[\hat{Q}_n] + [\theta_n] - [\beta_n] = [Q_n] \tag{7.6.7}$$

or, expanding as in Eq. (7.6.3*b*),

$$P_n U_1 + P_{n-1} U_2 + \ldots + P_{n-M+1} U_M + \theta_n - \beta_n = Q_n \qquad n = 1, \ldots, N \tag{7.6.8}$$

To ensure that the unit hydrograph represents one unit of direct runoff an additional

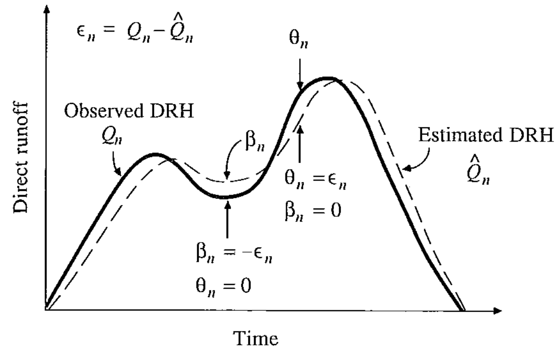


FIGURE 7.6.1
Deviation ϵ_n between observed and estimated direct runoff hydrographs is the sum of a positive deviation θ_n and a negative deviation β_n for solution by linear programming.

constraint equation is added:

$$\sum_{m=1}^{M} U_m = K \tag{7.6.9}$$

where K is a constant which converts the units of the ERH into the units of the DRH. Equations (7.6.6) to (7.6.9) constitute a linear program with *decision variables* (or unknowns) U_m, θ_n and β_n which may be solved using standard linear programming computer programs to produce the unit hydrograph. Linear programming requires all the decision variables to be non-negative, thereby ensuring the unit hydrograph ordinates will be non-negative.

The linear programming method developed in Example 7.6.1 is not limited in application to a single storm. Several ERHs and their resulting DRHs can be linked together as if they comprised one event and used to find a *composite unit hydrograph* best representing the response of the watershed to this set of storms. Multistorm analysis may also be carried out using the least-squares method (Diskin and Boneh, 1975; Mawdsley and Tagg, 1981).

In determination of the unit hydrograph from complex hydrographs, the abstractions are a significant source of error—although often assumed constant, the loss rate is actually a time-varying function whose value is affected by the moisture content of the watershed prior to the storm and by the storm pattern itself. Different unit hydrographs result from different assumptions about the pattern of losses. Newton and Vinyard (1967) account for errors in the loss rate by iteratively adjusting the ordinates of the ERH as well as those of the unit hydrograph so as to minimize the error in the DRH. Mays and Taur (1982) used nonlinear programming to simultaneously determine the loss rate for each storm period and the composite unit hydrograph ordinates for a multistorm event. Unver and Mays (1984) extended this nonlinear programming method to determine the optimal parameters for the loss-rate functions, and the composite unit hydrograph.

7.7 SYNTHETIC UNIT HYDROGRAPH

The unit hydrograph developed from rainfall and streamflow data on a watershed applies only for that watershed and for the point on the stream where the streamflow data were measured. Synthetic unit hydrograph procedures are used to develop unit hydrographs for other locations on the stream in the same watershed or for nearby watersheds of a similar character. There are three types of synthetic unit hydrographs: (1) those relating hydrograph characteristics (peak flow rate, base time, etc.) to watershed characteristics (Snyder, 1938; Gray, 1961), (2) those based on a dimensionless unit hydrograph (Soil Conservation Service, 1972), and (3) those based on models of watershed storage (Clark, 1943). Types (1) and (2) are described here and type (3) in Chap. 8.

Snyder's Synthetic Unit Hydrograph

In a study of watersheds located mainly in the Appalachian highlands of the United States, and varying in size from about 10 to 10,000 mi^2 (30 to 30,000 km^2), Snyder (1938) found synthetic relations for some characteristics of a *standard unit hydrograph* [Fig. 7.7.1*a*]. Additional such relations were found later (U.S. Army Corps of Engineers, 1959). These relations, in modified form are given below. From the relations, five characteristics of a *required unit hydrograph* [Fig. 7.7.1*b*] for a given excess rainfall duration may be calculated: the peak discharge per unit of watershed area, q_{pR}, the basin lag t_{pR} (time difference between the centroid of the excess rainfall hyetograph and the unit hydrograph peak), the base time t_b, and the widths W (in time units) of the unit hydrograph at 50 and 75 percent of the peak discharge. Using these characteristics the required unit hydrograph may be drawn. The variables are illustrated in Fig. 7.7.1.

Snyder defined a standard unit hydrograph as one whose rainfall duration t_r is related to the basin lag t_p by

$$t_p = 5.5t_r \tag{7.7.1}$$

For a standard unit hydrograph he found that:

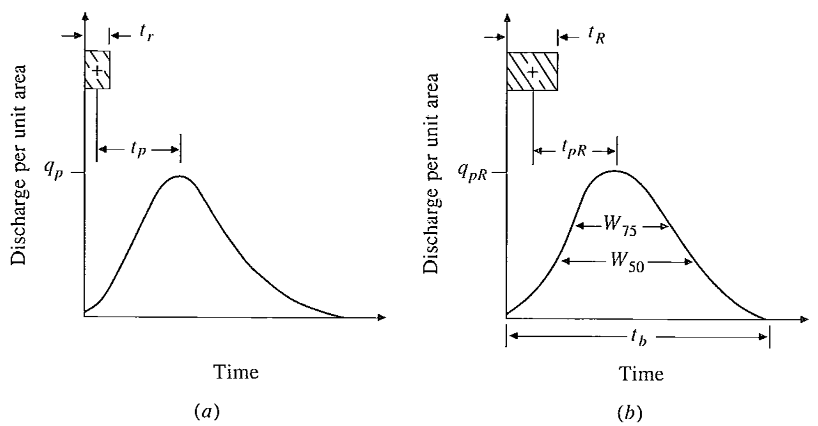


FIGURE 7.7.1
Snyder's synthetic unit hydrograph. (*a*) Standard unit hydrograph ($t_p = 5.5t_r$). (*b*) Required unit hydrograph ($t_{pR} \neq 5.5t_R$).

1. The basin lag is

$$t_p = C_1C_t(LL_c)^{0.3} \tag{7.7.2}$$

where t_p is in hours, L is the length of the main stream in kilometers (or miles) from the outlet to the upstream divide, L_c is the distance in kilometers (miles) from the outlet to a point on the stream nearest the centroid of the watershed area, $C_1 = 0.75$ (1.0 for the English system), and C_t is a coefficient derived from gaged watersheds in the same region.

2. The peak discharge per unit drainage area in $m^3/s{\cdot}km^2$ (cfs/mi^2) of the standard unit hydrograph is

$$q_p = \frac{C_2C_p}{t_p} \tag{7.7.3}$$

where $C_2 = 2.75$ (640 for the English system) and C_p is a coefficient derived from gaged watersheds in the same region.

To compute C_t and C_p for a gaged watershed, the values of L and L_c are measured from the basin map. From a *derived unit hydrograph* of the watershed are obtained values of its effective duration t_R in hours, its basin lag t_{pR} in hours, and its peak discharge per unit drainage area, q_{pR}, in $m^3/s{\cdot}km^2{\cdot}cm$ ($cfs/mi^2{\cdot}in$ for the English system). If $t_{pR} = 5.5t_R$, then $t_R = t_r$, $t_{pR} = t_p$, and $q_{pR} = q_p$, and C_t and C_p are computed by Eqs. (7.7.2) and (7.7.3). If t_{pR} is quite different from $5.5t_R$, the standard basin lag is

$$t_p = t_{pR} + \frac{t_r - t_R}{4} \tag{7.7.4}$$

and Eqs. (7.7.1) and (7.7.4) are solved simultaneously for t_r and t_p. The values of C_t and C_p are then computed from (7.7.2) and (7.7.3) with $q_{pR} = q_p$ and $t_{pR} = t_p$.

When an ungaged watershed appears to be similar to a gaged watershed, the coefficients C_t and C_p for the gaged watershed can be used in the above equations to derive the required synthetic unit hydrograph for the ungaged watershed.

3. The relationship between q_p and the peak discharge per unit drainage area q_{pR} of the required unit hydrograph is

$$q_{pR} = \frac{q_pt_p}{t_{pR}} \tag{7.7.5}$$

4. The base time t_b in hours of the unit hydrograph can be determined using the fact that the area under the unit hydrograph is equivalent to a direct runoff of 1 cm (1 inch in the English system). Assuming a triangular shape for the unit hydrograph, the base time may be estimated by

$$t_b = \frac{C_3}{q_{pR}} \tag{7.7.6}$$

where $C_3 = 5.56$ (1290 for the English system).

5. The width in hours of a unit hydrograph at a discharge equal to a certain percent of the peak discharge q_{pR} is given by

$$W = C_w q_{pR}^{-1.08} \tag{7.7.7}$$

where $C_w = 1.22$ (440 for English system) for the 75-percent width and 2.14 (770, English system) for the 50-percent width. Usually one-third of this width is distributed before the unit hydrograph peak time and two-thirds after the peak.

Example 7.7.1. From the basin map of a given watershed, the following quantities are measured: L = 150 km, L_c = 75 km, and drainage area = 3500 km^2. From the unit hydrograph derived for the watershed, the following are determined: t_R = 12 h, t_{pR} = 34 h, and peak discharge = 157.5 m^3/s·cm. Determine the coefficients C_t and C_p for the synthetic unit hydrograph of the watershed.

Solution. From the given data, $5.5t_R$ = 66 h, which is quite different from t_{pR} (34 h). Equation (7.7.4) yields

$$t_p = t_{pR} + \frac{t_r - t_R}{4}$$

$$= 34 + \frac{t_r - 12}{4} \tag{7.7.8}$$

Solving (7.7.1) and (7.7.8) simultaneously gives t_r = 5.9 h and t_p = 32.5 h. To calculate C_t, use (7.7.2):

$$t_p = C_1 C_t (LL_c)^{0.3}$$

$$32.5 = 0.75C_t(150 \times 75)^{0.3}$$

$$C_t = 2.65$$

The peak discharge per unit area is q_{pR} = 157.5/3500 = 0.045 m^3/s·km^2·cm. The coefficient C_p is calculated by Eq. (7.7.3) with $q_p = q_{pR}$, and $t_p = t_{pR}$:

$$q_{pR} = \frac{C_2 C_p}{t_{pR}}$$

$$0.045 = \frac{2.75C_p}{34.0}$$

$$C_p = 0.56$$

Example 7.7.2. Compute the six-hour synthetic unit hydrograph of a watershed having a drainage area of 2500 km^2 with L = 100 km and L_c = 50 km. This watershed is a sub-drainage area of the watershed in Example 7.7.1.

Solution. The values C_t = 2.64 and C_p = 0.56 determined in Example 7.7.1 can also be used for this watershed. Thus, Eq. (7.7.2) gives $t_p = 0.75 \times 2.64 \times (100 \times 50)^{0.3}$ = 25.5 h, and (7.7.1) gives t_r = 25.5/5.5 = 4.64 h. For a six-hour unit hydrograph, t_R = 6 h, and Eq. (7.7.4) gives $t_{pR} = t_p - (t_r - t_R)/4$ = 25.5 − (4.64−6)/4 = 25.8 h. Equation (7.7.3) gives $q_p = 2.75 \times 0.56/25.5 = 0.0604$

m^3/s·km^{2}cm and (7.7.5) gives $q_{pR} = 0.0604 \times 25.5/25.8 = 0.0597$ m^3/s·km^2·cm; the peak discharge is $0.0597 \times 2500 = 149.2$ m^3/s·cm. The widths of the unit hydrograph are given by Eq. (7.7.7). At 75 percent of peak discharge, $W = 1.22q_{pR}^{-1.08} = 1.22 \times 0.0597^{-1.08}$ = 25.6 h. A similar computation gives a W = 44.9 h at 50 percent of peak. The base time, given by Eq. (7.7.6), is $t_b = 5.56/q_{pR} = 5.56/0.0597$ = 93h. The hydrograph is drawn, as in Fig. 7.7.2, and checked to ensure that it represents a depth of direct runoff of 1 cm.

A further innovation in the use of Snyder's method has been the regionalization of unit hydrograph parameters. Espey, Altman and Graves (1977) developed a set of generalized equations for the construction of 10-minute unit hydrographs using a study of 41 watersheds ranging in size from 0.014 to 15 mi^2, and in impervious percentage from 2 to 100 percent. Of the 41 watersheds, 16 are located in Texas, 9 in North Carolina, 6 in Kentucky, 4 in Indiana, 2 each in Colorado and Mississippi, and 1 each in Tennessee and Pennsylvania. The equations are:

$$T_p = 3.1L^{0.23}S^{-0.25}I^{-0.18}\Phi^{1.57} \tag{7.7.9}$$

$$Q_p = 31.62 \times 10^3 A^{0.96} T_p^{-1.07} \tag{7.7.10}$$

$$T_B = 125.89 \times 10^3 A Q_p^{-0.95} \tag{7.7.11}$$

$$W_{50} = 16.22 \times 10^3 A^{0.93} Q_p^{-0.92} \tag{7.7.12}$$

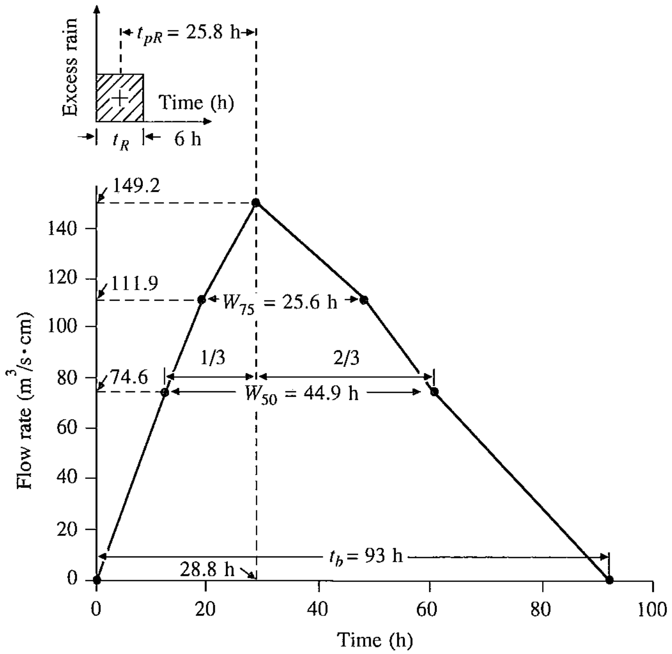


FIGURE 7.7.2
Synthetic unit hydrograph calculated by Snyder's method in Example 7.7.2

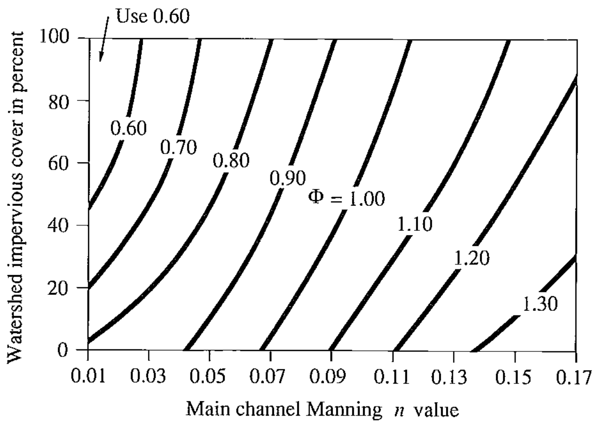


FIGURE 7.7.3
Watershed conveyance factor Φ as a function of channel roughness and watershed imperviousness. (Adapted with permission from Espey, Altman, and Graves, 1977.)

$$W_{75} = 3.24 \times 10^3 A^{0.79} Q_p^{-0.78} \tag{7.7.13}$$

where

L = the total distance (in feet) along the main channel from the point being considered to the upstream watershed boundary

S = the main channel slope (in feet per foot), defined by $H/0.8L$, where H is the difference in elevation between A and B. A is the point on the channel bottom at a distance of $0.2L$ downstream from the upstream watershed boundary; B is a point on the channel bottom at the downstream point being considered

I = the impervious area within the watershed (in percent), assumed equal to 5 percent for an undeveloped watershed

Φ = the dimensionless watershed conveyance factor, which is a function of percent impervious and roughness (Fig. 7.7.3)

A = the watershed drainage area (in square miles)

T_p = the time of rise to the peak of the unit hydrograph from the beginning of runoff (in minutes)

Q_p = the peak flow of the unit hydrograph (in cfs/in)

T_B = the time base of the unit hydrograph (in minutes)

W_{50} = the width of the hydrograph at 50 percent of Q_p (in minutes)

W_{75} = the width of at 75 percent of Q_p (in minutes)

SCS Dimensionless Hydrograph

The SCS dimensionless hydrograph is a synthetic unit hydrograph in which the discharge is expressed by the ratio of discharge q to peak discharge q_p and the

time by the ratio of time t to the time of rise of the unit hydrograph, T_p. Given the peak discharge and lag time for the duration of excess rainfall, the unit hydrograph can be estimated from the synthetic dimensionless hydrograph for the given basin. Figure 7.7.4(*a*) shows such a dimensionless hydrograph, prepared from the unit hydrographs of a variety of watersheds. The values of q_p and T_p may be estimated using a simplified model of a triangular unit hydrograph as shown in Figure 7.7.4(*b*), where the time is in hours and the discharge in $m^3/s \cdot cm$ (or cfs/in) (Soil Conservation Service, 1972).

From a review of a large number of unit hydrographs, the Soil Conservation Service suggests the time of recession may be approximated as $1.67\ T_p$. As the area under the unit hydrograph should be equal to a direct runoff of 1 cm (or 1 in), it can be shown that

$$q_p = \frac{CA}{T_p} \tag{7.7.14}$$

where $C = 2.08$ (483.4 in the English system) and A is the drainage area in square kilometers (square miles).

Further, a study of unit hydrographs of many large and small rural watersheds indicates that the basin lag $t_p \simeq 0.6T_c$, where T_c is the time of concentration of the watershed. As shown in Fig. 7.7.4(*b*), time of rise T_p can be expressed in terms of lag time t_p and the duration of effective rainfall t_r

$$T_p = \frac{t_r}{2} + t_p \tag{7.7.15}$$

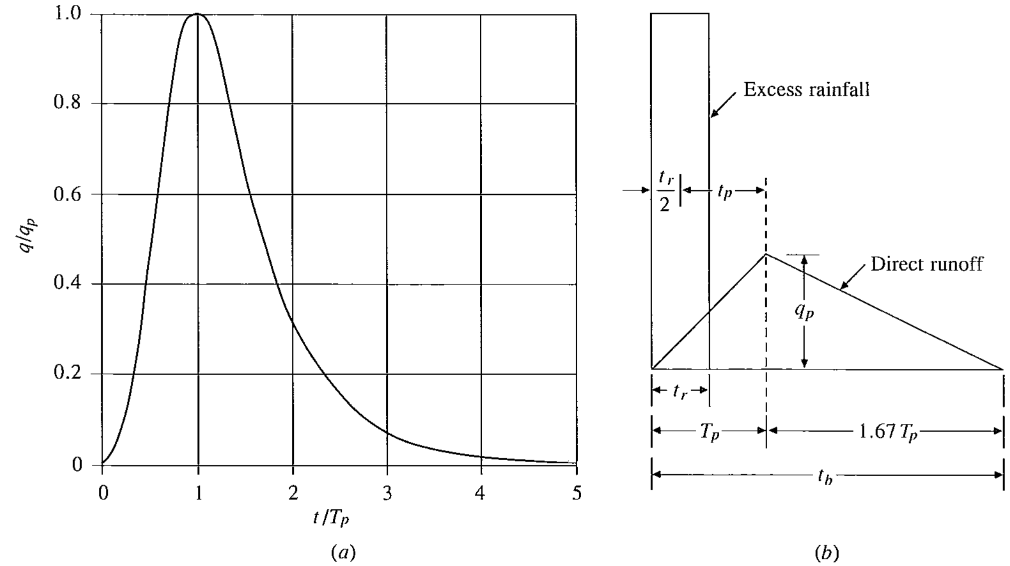


FIGURE 7.7.4
Soil Conservation Service synthetic unit hydrographs (*a*) Dimensionless hydrograph and (*b*) triangular unit hydrograph. (*Source*: Soil Conservation Service, 1972.)

Example 7.7.3. Construct a 10-minute SCS unit hydrograph for a basin of area 3.0 km^2 and time of concentration 1.25 h.

Solution. The duration t_r = 10 min =0.166 h, lag time $t_p = 0.6T_c = 0.6 \times 1.25$ = 0.75 h, and rise time $T_p = t_r/2 + t_p = 0.166/2 + 0.75 = 0.833$ h. From Eq. (7.7.14), $q_p = 2.08 \times 3.0/0.833 = 7.49$ m^3/s·cm. The dimensionless hydrograph in Fig. 7.7.4 may be converted to the required dimensions by multiplying the values on the horizontal axis by T_p and those on the vertical axis by q_p. Alternatively, the triangular unit hydrograph can be drawn with $t_b = 2.67T_p = 2.22$ h. The depth of direct runoff is checked to equal 1 cm.

7.8 UNIT HYDROGRAPHS FOR DIFFERENT RAINFALL DURATIONS

When a unit hydrograph of a given excess-rainfall duration is available, the unit hydrographs of other durations can be derived. If other durations are integral multiples of the given duration, the new unit hydrograph can be easily computed by application of the principles of superposition and proportionality. However, a general method of derivation applicable to unit hydrographs of any required duration may be used on the basis of the principle of superposition. This is the *S-hydrograph method.*

The theoretical *S-hydrograph* is that resulting from a continuous excess rainfall at a constant rate of 1 cm/h (or 1 in/h) for an indefinite period. This is the unit step response function of a watershed system. The curve assumes a deformed S shape and its ordinates ultimately approach the rate of excess rainfall at a time of equilibrium. This step response function $g(t)$ can be derived from the unit pulse response function $h(t)$ of the unit hydrograph, as follows.

From Eq. (7.2.4), the response at time t to a unit pulse of duration Δt beginning at time 0 is

$$h(t) = \frac{1}{\Delta t}[g(t) - g(t - \Delta t)] \qquad (7.8.1)$$

Similarly, the response at time t to a unit pulse beginning at time Δt is equal to $h(t - \Delta t)$, that is, $h(t)$ lagged by Δt time units:

$$h(t - \Delta t) = \frac{1}{\Delta t}[g(t - \Delta t) - g(t - 2\,\Delta t)] \qquad (7.8.2)$$

and the response at time t to a third unit pulse beginning at time $2\Delta t$ is

$$h(t - 2\Delta t) = \frac{1}{\Delta t}[g(t - 2\Delta t) - g(t - 3\,\Delta t)] \qquad (7.8.3)$$

Continuing this process indefinitely, summing the resulting equations, and rearranging, yields the unit step response function, or S-hydrograph, as shown in Fig. 7.8.1(a):

$$g(t) = \Delta t\,[h(t) + h(t - \Delta t) + h(t - 2\,\Delta t) + \ldots] \qquad (7.8.4)$$

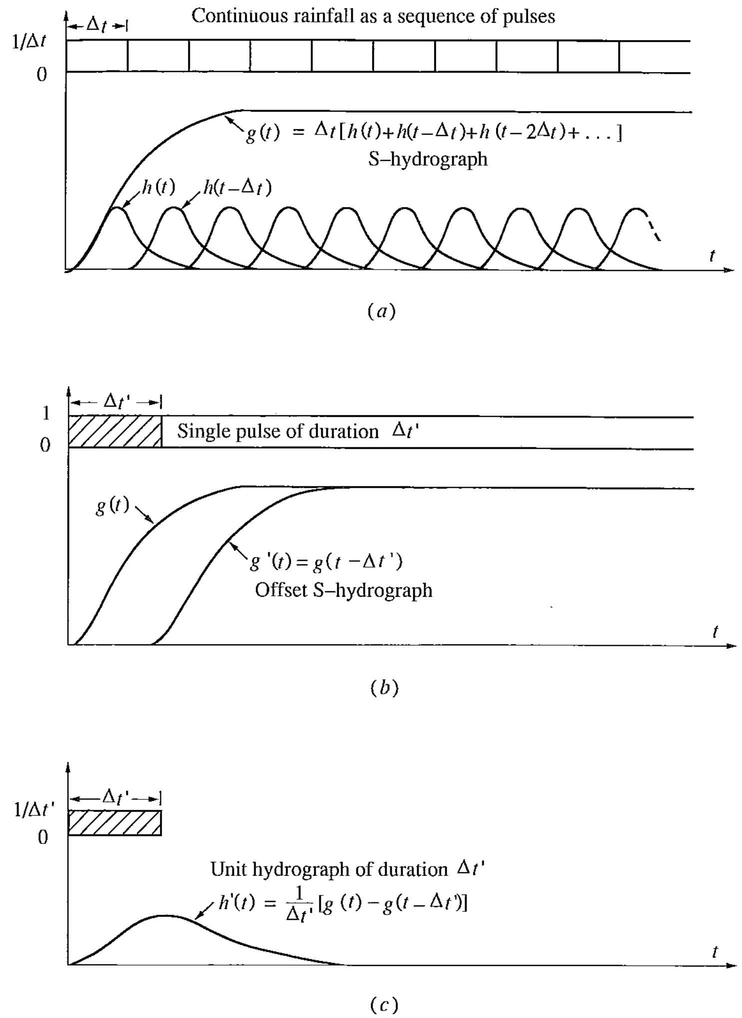


FIGURE 7.8.1
Using the S-hydrograph to find a unit hydrograph of duration $\Delta t'$ from a unit hydrograph of duration Δt.

where the summation is multiplied by Δt so that $g(t)$ will correspond to an input rate of 1, rather than $1/\Delta t$ as used for each of the unit pulses.

Theoretically, the S-hydrograph so derived should be a smooth curve, because the input excess rainfall is assumed to be at a constant, continuous rate. However, the summation process will result in an undulatory form if there are errors in the rainfall abstractions or baseflow separation, or if the actual duration of excess rainfall is not the derived duration for the unit hydrograph. A duration which produces minimum undulation can be found by trial. Undulation of the curve may be also caused by nonuniform temporal and areal distribution of rainfall; furthermore, when the natural data are not linear, the resulting unstable system oscillations may produce negative ordinates. In such cases, an optimization technique may be used to obtain a smoother unit hydrograph.

After the S-hydrograph is constructed, the unit hydrograph of a given duration can be derived as follows: Advance, or offset, the position of the S-hydrograph by a period equal to the desired duration $\Delta t'$ and call this S-hydrograph an *offset S-hydrograph,* $g'(t)$ [Fig. 7.8.1(*b*)], defined by

$$g'(t) = g(t - \Delta t') \tag{7.8.5}$$

The difference between the ordinates of the original S-hydrograph and the offset S-hydrograph, divided by $\Delta t'$, gives the desired unit hydrograph [Fig. 7.8.1(*c*)]:

$$h'(t) = \frac{1}{\Delta t'}[g(t) - g(t - \Delta t')] \tag{7.8.6}$$

Example 7.8.1. Use the 0.5-hour unit hydrograph in Table 7.4.3 (from Example 7.4.1) to produce the S-hydrograph and the 1.5-h unit hydrograph for this watershed.

Solution. The 0.5-h unit hydrograph is shown in column 2 of Table 7.8.1. The S-hydrograph is found using (7.8.4) with $\Delta t = 0.5$ h. For $t = 0.5$ h, $g(t) = \Delta t\, h(t) = 0.5 \times 404 = 202$ cfs; for $t = 1$ h, $g(t) = \Delta t[h(t) + h(t-0.5)] = 0.5 \times (1079 + 404) = 742$ cfs; for $t = 1.5$ h, $g(t) = \Delta t[h(t) + h(t-0.5) + h(t-1.0)] = 0.5 \times (2343 + 1079 + 404) = 1913$ cfs; and so on, as shown in column 3 of Table 7.8.1. The S-hydrograph is offset by $\Delta t' = 1.5$ h (column 4) to give $g(t - \Delta t')$, and the difference divided by $\Delta t'$ to give the 1.5-h unit hydrograph $h'(t)$ (column 5). For example, for $t = 2.0$ h, $h(t) = (3166 - 202)/1.5 = 1976$ cfs/in.

TABLE 7.8.1
Calculation of a 1.5-h unit hydrograph by the S-hydrograph method (Example 7.8.1)

1 Time t (h)	2 0.5-h unit hydrograph $h(t)$ (cfs/in)	3 S-hydrograph $g(t)$ (cfs)	4 Lagged S-hydrograph $g(t - \Delta t')$ (cfs)	5 1.5-h unit hydrograph $h'(t)$ (cfs/in)
0.5	404	202	0	135
1.0	1079	742	0	495
1.5	2343	1913	0	1275
2.0	2506	3166	202	1976
2.5	1460	3896	742	2103
3.0	453	4123	1913	1473
3.5	381	4313	3166	765
4.0	274	4450	3896	369
4.5	173	4537	4123	276
5.0	0	4537	4313	149
5.5	0	4537	4450	58
6.0	0	4537	4537	0

Instantaneous Unit Hydrograph

If the excess rainfall is of unit amount and its duration is infinitesimally small, the resulting hydrograph is an impulse response function (Sec. 7.2) called the instantaneous unit hydrograph (IUH). For an IUH, the excess rainfall is applied to the drainage area in zero time. Of course, this is only a theoretical concept and cannot be realized in actual watersheds, but it is useful because the IUH characterizes the watershed's response to rainfall without reference to the rainfall duration. Therefore, the IUH can be related to watershed geomorphology (Rodriguez-Iturbe and Valdes, 1979; Gupta, Waymire, and Wang, 1980).

The convolution integral (7.2.1) is

$$Q(t) = \int_0^t u(t - \tau) I(\tau)\, d\tau \tag{7.8.7}$$

If the quantities $I(\tau)$ and $Q(t)$ have the same dimensions, the ordinate of the IUH must have dimensions $[T^{-1}]$. The properties of the IUH are as follows, with $l = t - \tau$:

$$\begin{aligned} 0 \le u(l) &\le \text{some positive peak value} && \text{for } l > 0 \\ u(l) &= 0 && \text{for } l \le 0 \\ u(l) &\to 0 && \text{as } l \to \infty \end{aligned} \tag{7.8.8}$$

$$\int_0^\infty u(l)\, dl = 1 \qquad \text{and} \qquad \int_0^\infty u(l)\, l\, dl = t_L$$

The quantity t_L is the lag time of the IUH. It can be shown that t_L gives the time interval between the centroid of an excess rainfall hyetograph and that of the corresponding direct runoff hydrograph. Note the difference between t_L and the variable t_p used for synthetic unit hydrograph lag time—t_p measures the time from the centroid of the excess rainfall to the peak, not the centroid, of the direct runoff hydrograph. The ideal shape of an IUH as described above resembles that of a single-peaked direct-runoff hydrograph, however, an IUH can have negative and undulating ordinates.

There are several methods to determine an IUH from a given ERH and DRH. For an approximation, the IUH ordinate at time t is simply set equal to the slope at time t of an S-hydrograph constructed for an excess rainfall intensity of unit depth per unit time. This procedure is based on the fact that the S-hydrograph is an integral curve of the IUH; that is, its ordinate at time t is equal to the integral of the area under the IUH from 0 to t. The IUH so obtained is in general only an approximation because the slope of an S-hydrograph is difficult to measure accurately.

The IUH can be determined by various methods of mathematical inversion, using, for example, orthogonal functions such as Fourier series (O'Donnell, 1960) or Laguerre functions (Dooge, 1973); integral transforms such as the Laplace transform (Chow, 1964), the Fourier transform (Blank, Delleur, and Giorgini, 1971), and the Z transform (Bree, 1978); and mathematical modeling related to watershed geomorphology (Sec. 8.5).

REFERENCES

Blank, D., J. W. Delleur, and A. Giorgini, Oscillatory kernel functions in linear hydrologic models, *Water Resour. Res.*, vol. 7, no. 5, pp. 1102–1117, 1971.

Bradley, S. P., A. C. Hax, and T. L. Magnanti, *Applied Mathematical Programming,* Addison-Wesley, Reading, Mass., 1977.

Bree, T., The stability of parameter estimation in the general linear model, *J. Hydrol.*, vol. 37, no. 1/2, pp. 47–66, 1978.

Chow, V. T., Runoff, in *Handbook of Applied Hydrology,* sec. 14, pp. 14-24 to 14-27, McGraw-Hill, New York, 1964.

Chow, V. T., and V. C. Kulandaiswamy, General hydrologic system model, *J. Hyd. Div., Am. Soc. Civ. Eng.*, vol. 97, no. HY6, pp. 791–804, 1971.

Clark, C. O., Storage and the unit hydrograph, *Proc. Am. Soc. Civ. Eng.*, vol. 9, pp. 1333–1360, 1943.

Collins, W. T., Runoff distribution graphs from precipitation occurring in more than one time unit, *Civ. Eng.*, vol. 9, no. 9, pp. 559–561, 1939.

Deininger, R. A., Linear programming for hydrologic analyses, *Water Resour. Res.*, vol. 5, no. 5, pp. 1105–1109, 1969.

Diskin, M. H., and A. Boneh, Determination of an optimal IUH for linear time invariant systems from the multi-storm records, *J. Hydrol.*, vol. 24, pp. 57–76, 1975.

Dooge, J. C. I., Linear theory of hydrologic systems, *Tech. Bull. No. 1468, Agric. Res. Serv.*, pp. 117–124, October, U. S. Department of Agriculture, Washington, D. C., 1973.

Eagleson, P. S., R. Mejia-R., and F. March, Computation of optimum realizable unit hydrographs, *Water Resour. Res.*, vol. 2, no. 4, pp. 755–764, 1966.

Espey, W. H., Jr., D. G. Altman, and C. B. Graves, Nomographs for ten-minute unit hydrographs for small urban watersheds, *Tech. Memo. No. 32, Urban Water Resources Research Prog., Am. Soc. Civ. Eng.*, New York, Dec. 1977

Gray, D. M., Synthetic unit hydrographs for small watersheds, *J. Hyd. Div., Am. Soc. Civ. Eng.*, vol. 87, no. HY4, pp. 33–54, 1961.

Gupta, V. K., E. Waymire, and C. T. Wang, A representation of an instantaneous unit hydrograph from geomorphology, *Water Resour. Res.*, vol. 16, no. 5, pp. 855–862, 1980.

Heerdegen, R. G., The unit hydrograph: a satisfactory model of watershed response? *Water Resour. Bul.* vol. 10, no. 6, pp. 1143–1161, 1974.

Hillier, F. S., and G. J. Lieberman, *Operations Research,* 2nd ed., Holden Day Inc., San Francisco, Calif., 1974.

Kreyszig, E., *Advanced Engineering Mathematics,* Wiley, New York, 1968.

Linsley, R. K., M. A. Kohler, and J. L. H. Paulhus, *Hydrology for engineers,* 3rd ed., McGraw-Hill, New York, 1982.

Mawdsley, J. A., and A. F. Tagg, Identification of unit hydrographs from multi-event analysis, *J. Hydrol.*, vol. 49, no. 3/4, pp. 315–327, 1981.

Mays, L. W., and S. L. Coles, Optimization of unit hydrograph determination, *J. Hyd. Div., Am. Soc. Civ. Eng.*, vol. 106, no. HY1, pp. 85–97, 1980.

Mays, L. W., and C. K. Taur, Unit hydrographs via nonlinear programming, *Water Resour. Res.*, vol. 18, no. 4, pp. 744–752, 1982.

Newton, D. S., and J. W. Vinyard, Computer-determined unit hydrograph from floods, *J. Hyd, Div., Am. Soc. Civ. Eng.*, vol. 93, no. HY5, pp. 219–235, 1967.

O'Donnell, T., Instantaneous unit hydrograph by harmonic analysis, *Proc. Gen. Ass. Helsinki, IASH,* publ. 51, pp. 546–557, 1960.

Rodriguez-Iturbe, I., and J. B. Valdes, The geomorphologic structure of hydrologic response, *Water Resour. Res.*, vol. 15, no. 6, pp. 1409–1420, 1979.

Sherman, L. K., Streamflow from rainfall by the unit-graph method, *Eng. News Rec.*, vol. 108, pp. 501–505, April 7, 1932.

Singh, K. P., Unit hydrographs—a comparative study, *Water Resour. Bul.*, vol. 12, no. 2, pp. 381–392, April, 1976.

Snyder, W. M., Hydrograph analysis by the method of least squares, *Proc. Amer. Soc. Civ. Eng.*, vol. 81, 1–24, 1955.

Snyder, F. F., Synthetic unit-graphs, *Trans. Am. Geophys. Union,* vol. 19, pp. 447–454, 1938.

Soil Conservation Service, Hydrology, sec. 4 of *National Engineering Handbook,* Soil Conservation Service, U. S. Department of Agriculture, Washington, D.C., 1972.

Unver, O., and L. W. Mays, Optimal determination of loss rate functions and unit hydrographs, *Water Resour. Res.*, vol. 20, no. 2, pp. 203–214, 1984.

U. S. Army Corps of Engineers, Flood hydrograph analysis and computations, *Engineering and Design Manual, EM 1110-2-1405,* U. S. Government Printing Office, Washington, D.C., August 31, 1959.

BIBLIOGRAPHY

Amorocho, J., Discussion of predicting storm runoff on small experimental watersheds by N.E. Minshall, *J. Hyd. Div., Am. Soc. Civ. Eng.*, vol. 87, no. HY2, pp. 185–191, 1961.

Amorocho, J., Nonlinear hydrologic analysis, in *Advances in Hydroscience,* ed. by V. T. Chow, Academic Press, New York, vol. 9, pp. 203–251, 1973.

Amorocho, J., and A. Brandstetter, Determination of nonlinear functions in rainfall-runoff processes, *Water Resour. Res.*, vol. 7, no. 1, pp. 1087–1101, 1971.

Bidwell, V. J., Regression analysis of nonlinear catchment systems, *Water Resour. Res.*, vol. 7, no. 5, pp. 1118–1126, 1971.

Broome, P., and R. H. Spigel, A linear model of storm runoff from some urban catchments in New Zealand, *J. Hydrol.* (New Zealand), vol. 21, no. 1, pp. 13–33, 1982.

Croley, T. E., II, Gamma synthetic hydrographs, *J. Hydrol.*, vol. 47, pp. 41–52, 1980.

Diskin, M. H., On the derivation of linkage equations for Laguerre function coefficients, *J. Hydrol.*, vol. 32, pp. 321–327, 1977.

Diskin, M. H., and A. Boneh, The kernel function of linear nonstationary surface runoff systems, *Water Resour. Res.*, vol. 10, no. 4, pp. 753–761, 1974.

Gray, D. M., Synthetic unit hydrographs for small watersheds, *J. Hyd. Div., Am. Soc. Civ. Eng.*, vol. 87, no. HY4, pp. 33–54, 1961.

Hall, M. J., On the smoothing of oscillations in finite-period unit hydrographs derived by the harmonic method, *Hydrol. Sci. Bull.*, vol. 22, no. 2, pp. 313–324, 1977.

Hino, M., and K. Nadaoka, Mathematical derivation of linear and nonlinear runoff kernels, *Water Resour. Res.*, vol. 15, no. 4, pp. 918–928, 1979.

Hossain, A., A. R. Rao, and J. W. Delleur, Estimation of direct runoff from urban watersheds, *J. Hyd. Div., Am. Soc. Civ. Eng.*, vol. 104, no. HY2, pp. 169–188, 1978.

Jacoby, S., A mathematical model for nonlinear hydrologic systems, *J. Geophys. Res.*, vol. 71, no. 20, pp. 4811–4824, 1966.

Levin, A. G., Flash flood forecasts with hydrograph separation by individual time intervals, *Sov. Hydrol.*, vol. 18, no. 1, pp. 40–45, 1979.

Minshall, N. E., Predicting storm runoff on small experimental watersheds, *J. Hyd. Div., Am. Soc. Civ. Eng.*, vol. 86, no. HY8, pp. 17–38, 1960.

Naef, F., Can we model the rainfall-runoff process today? *Hydrol. Sci. Bull.*, vol. 26, no. 3, pp. 281–289, 1981.

Neuman, S. P., and G. de Marsily, Identification of linear systems response by parametric programming, *Water Resour. Res.*, vol. 12, no. 2, pp. 253–262, 1976.

O'Connor, K. M., A discrete linear cascade model for hydrology, *J. Hydrol.*, vol. 29, pp. 203–242, 1976.
Papazafiriou, Z. G., Polynomial approximation of the kernels of closed linear hydrologic systems, *J. Hydrol.*, vol. 27, pp. 319–329, 1975.
Papazafiriou, Z. G., Linear and nonlinear approaches for short-term runoff estimations in time-invariant open hydrologic systems, *J. Hydrol.*, vol. 30, pp. 63–80, 1976.
Raudkivi, A. J., *Hydrology*, Chap. 7, Pergamon Press, Oxford, 1979.
Rusin, S. A., Nonstationary model of the transformation of effective precipitation into runoff, *Sov. Hydrol.*, vol. 4, pp. 289–300, 1973.
Sarma, P. B. S., J. W. Delleur, and A. R. Rao, Comparison of rainfall-runoff models for urban areas, *J. Hydrol.*, vol. 18, pp. 329–347, 1973.
Viessman, W., Jr., J. W. Knapp, G. L. Lewis, and T. E. Harbaugh, *Introduction to Hydrology*, 2nd ed., Chap. 4, Harper and Row, New York, 1977.
Zand, S. M., and J. A. Harder, Application of nonlinear system identification to the lower Mekong River, Southeast Asia, *Water Resour. Res.*, vol. 9, no. 2, pp. 290–297, 1973.

PROBLEMS

7.2.1 A system has a discrete pulse response function with ordinates 0.1, 0.5, 0.3, and 0.1 units. Calculate the output from this system if it has a pulse input of (*a*) 3 units, (*b*) 4 units, (*c*) 3 units in the first time interval followed by 4 units in the second.

7.2.2 A system has the following unit pulse response function: 0.27, 0.36, 0.18, 0.09, 0.05, 0.03, 0.01, 0.01. Calculate the output from this system if it has input (*a*) 2 units, (*b*) 3 units, (*c*) 2 units in the first time interval followed by 3 units in the second time interval.

7.2.3 Calculate and plot the impulse response function $u(t)$, the step response function $g(t)$, the continuous pulse response function $h(t)$, and the discrete pulse response function U_n for a linear reservoir having $k = 1$ h and $\Delta t = 2$ h.

7.2.4 A watershed is modeled as a linear reservoir with $k = 1$ h. Calculate its impulse response function and its pulse response functions for unit pulses of durations 0.5, 1.0, 1.5 and 2.0 h. Plot the response functions for $0 < t < 6$ h.

7.2.5 A watershed modeled as a linear reservoir with $k = 3$ h receives 3 in of excess rainfall in the first two hours of a storm and 2 in of excess rainfall in the second two hours. Calculate the direct runoff hydrograph from this watershed.

7.2.6 Show that the lag time t_L between the centroids of the excess rainfall hyetograph and the direct runoff hydrograph is equal to the storage constant k for a watershed modeled as a linear reservoir.

7.3.1 A watershed has a drainage area of 450 km^2, and its three-hour unit hydrograph has a peak discharge of 150 m^3/s·cm. For English units, what is the peak discharge in cfs/in of the three-hour unit hydrograph?

7.4.1 The excess rainfall and direct runoff recorded for a storm are as follows:

Time (h)	1	2	3	4	5	6	7	8	9
Excess rainfall (in)	1.0	2.0		1.0					
Direct runoff (cfs)	10	120	400	560	500	450	250	100	50

Calculate the one-hour unit hydrograph.

7.4.2 What is the area of the watershed in Prob. 7.4.1?

7.4.3 Derive by deconvolution the six-hour unit hydrograph from the following data for a watershed having a drainage area of 216 km^2, assuming a constant rainfall abstraction rate and a constant baseflow of 20 m^3/s.

Six-hour period	1	2	3	4	5	6	7	8	9	10	11
Rainfall (cm)	1.5	3.5	2.5	1.5							
Streamflow (m^3/s)	26	71	174	226	173	99	49	33	26	22	21

7.4.4 Given below is the flood hydrograph from a storm on a drainage area of 2.5 mi^2.

Hour	1	2	3	4	5	6	7
Discharge (cfs)	52	48	44	203	816	1122	1138
Hour	8	9	10	11	12	13	
Discharge (cfs)	685	327	158	65	47	34	

Excess rainfall of nearly uniform intensity occurred continuously during the fourth, fifth, and sixth hours. Baseflow separation is accomplished by plotting the logarithm of the discharge against time. During the rising flood, the logarithm of baseflow follows a straight line with slope determined from the flow in hours 1–3. From the point of inflection of the falling limb of the flood hydrograph (hour 8), the logarithm of baseflow follows a straight line with slope determined from the flow in hours 11–13. Between the peak of the flood hydrograph and the point of inflection, the logarithm of baseflow is assumed to vary linearly. Derive the one-hour unit hydrograph by deconvolution.

7.4.5 An intense storm with approximately constant intensity lasting six hours over a watershed of area 785 km^2 produced the following discharges Q in m^3/s:

Hour	0	2	4	6	8	10	12	14	16	18	20
Q	18	21	28	44	70	118	228	342	413	393	334
Q_b	18	20	25	32	40	47	54	61	68	75	79
Hour	22	24	26	28	30	32	34	36	38	40	
Q	270	216	171	138	113	97	84	75	66	59	
Q_b	77	73	69	66	63	60	57	55	52	49	
Hour	42	44	46	48	50	52	54	56	58	60	
Q	54	49	46	42	40	38	36	34	33	33	
Q_b	47	44	42	40	38	37	35	34	33	33	

The baseflow Q_b has been estimated from the appearance of the observed hydrograph. Use deconvolution to determine the two-hour unit hydrograph.

7.5.1 Use the unit hydrograph developed in Prob. 7.4.3 to calculate the streamflow hydrograph from a 12-hour-duration storm having 2 cm of rainfall excess in the first six hours and 3 cm in the second six hours. Assume a constant baseflow rate of 30 m^3/s.

7.5.2 Use the one-hour unit hydrograph developed in Prob. 7.4.4 to calculate the streamflow hydrograph for a three-hour storm with a uniform rainfall intensity of 3 in/h. Assume abstractions are constant at 0.5 in/h and baseflow is the same as determined in Prob. 7.4.4.

7.5.3 Use the two-hour unit hydrograph determined in Prob. 7.4.5 to calculate the streamflow hydrograph from a four-hour storm in which 5 cm of excess rainfall fell in the first two hours and 6 cm in the second two hours. Assume the same baseflow rate as given in Prob. 7.4.5.

7.5.4 The six-hour unit hydrograph of a watershed having a drainage area equal to 393 km^2 is as follows:

Time (h)	0	6	12	18	24	30	36	42
Unit hydrograph (m^3/s·cm)	0	1.8	30.9	85.6	41.8	14.6	5.5	1.8

For a storm over the watershed having excess rainfall of 5 cm for the first six hours and 15 cm for the second six hours, compute the streamflow hydrograph, assuming constant baseflow of 100 m^3/s.

7.5.5 The one-hour unit hydrograph for a watershed is given below. Determine the runoff from this watershed for the storm pattern given. The abstractions have a constant rate of 0.3 in/h. What is the area of this watershed?

Time (h)	1	2	3	4	5	6
Precipitation (in)	0.5	1.0	1.5	0.5		
Unit hydrograph (cfs/in)	10	100	200	150	100	50

7.5.6 Use the same unit hydrograph as in Prob. 7.5.5 and determine the direct runoff hydrograph for a two-hour storm with 1 in of excess rainfall the first hour and 2 in the second hour. What is the area of this watershed?

7.5.7 An agricultural watershed was urbanized over a period of 20 years. A triangular unit hydrograph was developed for this watershed for an excess rainfall duration of one hour. Before urbanization, the average rate of infiltration and other losses was 0.30 in/h, and the unit hydrograph had a peak discharge of 400 cfs/in at 3 h and a base time of 9 h. After urbanization, because of the increase in impervious surfaces, the loss rate dropped to 0.15 in/h, the peak discharge of the unit hydrograph was increased to 600 cfs/in, occurring at 1 h, and the base time was reduced to 6 h. For a two-hour storm in which 1.0 in of rain fell the first hour and 0.50 in the second hour, determine the direct runoff hydrographs before and after urbanization.

7.5.8 The ordinates at one-hour intervals of a one-hour unit hydrograph are (in cfs/in): 269, 538, 807, 645, 484, 323, and 161. Calculate the direct runoff hydrograph from a two-hour storm in which 4 in of excess rainfall occurs at a constant rate. What is the watershed area (mi^2)?

7.5.9 The 10-minute triangular unit hydrograph from a watershed has a peak discharge of 100 cfs/in at 40 min and a total duration of 100 min. Calculate the streamflow hydrograph from this watershed for a storm in which 2 in of rain falls in the first 10 minutes and 1 in in the second 10 minutes, assuming that the loss rate is $\phi = 0.6$ in/h and the baseflow rate is 20 cfs.

7.6.1 The July 19–20, 1979, storm on the Shoal Creek watershed at Northwest Park in Austin, Texas, resulted in the following rainfall-runoff values.

Time(h)	0.5	1.0	1.5	2.0	2.5	3.0	3.5
Rainfall (in)	1.17	0.32	0.305	0.67	0.545	0.10	0.06
Direct runoff (cfs)	11.0	372.0	440.0	506.0	2110.0	1077.0	429.3
Time (h)	4.0	4.5	5.0	5.5	6.0	6.5	7.0
Direct runoff (cfs)	226.6	119.0	64.7	39.7	28.0	21.7	16.7
Time (h)	7.5	8.0	8.5	9.0			
Direct runoff (cfs)	13.3	9.2	9.0	7.3			

Determine the half-hour unit hydrograph using linear programming. Assume that a uniform loss rate is valid. The watershed area is 7.03 mi^2. Compare the unit hydrograph with that determined in Example 7.4.1 for this watershed.

7.6.2 A storm on April 16, 1977, on the Shoal Creek watershed at Northwest Park in Austin, Texas, resulted in the following rainfall-runoff values:

Time (h)	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
Rainfall (in)	0.28	0.12	0.13	0.14	0.18	0.14	0.07		
Direct runoff (cfs)	32	67	121	189	279	290	237	160	108
Time (h)	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0
Direct runoff (cfs)	72	54	44	33	28	22	20	18	16

Determine the half-hour unit hydrograph by linear programming. Assume that a uniform loss rate is valid. The watershed area is 7.03 mi^2. Compare the unit hydrograph with that developed in Example 7.4.1 for this watershed.

7.6.3 Combine the data from Probs. 7.6.1 and 7.6.2 and calculate a composite unit hydrograph from this watershed by linear programming. Compare the composite unit hydrograph with those determined from the individual storms.

7.6.4 Solve Prob. 7.6.1 by linear regression.

7.6.5 Solve Prob. 7.6.2 by linear regression.

7.6.6 Solve Prob. 7.6.3 by linear regression.

7.7.1 The City of Austin, Texas, uses generalized equations (7.7.9)–(7.7.13) to determine the parameters for 10-minute-duration unit hydrographs for small watersheds. Determine the 10-minute unit hydrographs for levels of imperviousness 10, 40, and 70 percent, on a watershed that has an area of 0.42 mi^2 with a main channel length of 5760 ft. The main channel slope is 0.015 ft/ft as defined in Sec. 7.7. Assume $\Phi = 0.8$. Plot the three unit hydrographs on the same graph.

7.7.2 Using the 10-minute unit hydrograph equations (7.7.9)–(7.7.13), develop the unit hydrograph for a small watershed of 0.3 mi^2 that has a main channel slope of 0.009 ft/ft. The main channel area is 2000 feet long and the percent imperviousness is 25. Next, develop the 10-minute unit hydrograph for the same watershed assuming the main channel length is 6000 feet long. Plot and compare the two unit hydrographs. Assume $n = 0.05$ for the main channel.

7.7.3 Determine direct runoff hydrographs using the two 10-minute unit hydrographs derived in the previous problem for the watersheds with main channel lengths of 2000 ft and 6000 ft. Consider a storm having 1.2 inches rainfall uniformly distributed over the first 30 minutes and 1.5 inches in the second 30 minutes. The infiltration losses are to be determined using the SCS method described in Chap. 5 for curve number CN = 85.

7.7.4 The 10-minute unit hydrograph for a 0.86-mi^2 watershed has 10-minute ordinates in cfs/in of 134, 392, 475, 397, 329, 273, 227, 188, 156, 129, 107, 89, 74, 61, 51, 42, 35, 29, 24, 10, 17, 14, 11, ... Determine the peaking coefficient C_p for Snyder's method. The main channel length is 10,500 ft, and $L_c = 6000$ ft. Determine the coefficient C_t.

7.7.5 Several equations for computing basin lag have been reported in the literature. One such equation that also considers the basin slope was presented by Linsley, Kohler, and Paulhus (1982):

$$t_p = C_t\left(\frac{LL_c}{\sqrt{S}}\right)^n$$

For a basin slope of $S = 0.008$ and $n = 0.4$, determine the coefficient C_t for the unit hydrograph in the previous problem.

7.7.6 The following information for watershed A and its two-hour unit hydrograph has been determined: area = 100 mi^2, $L_c = 10$ mi, $L = 24$ mi, $t_R = 2$ h, $t_{pR} = 6$ h, $Q_p = 9750$ cfs/in, $W_{50} = 4.1$ h, and $W_{75} = 2$ h. Watershed B, which is assumed to be hydrologically similar to watershed A, has the following characteristics: area = 70 mi^2, $L = 15.6$ mi, and $L_c = 9.4$ mi. Determine the one-hour synthetic unit hydrograph for watershed B.

7.7.7 (*a*) Determine the coefficients C_p and C_t for a watershed of area 100 mi^2 with $L = 20$ mi and $L_c = 12$ mi, for $t_R = 2$ h and $t_{pR} = 5$ h. The peak of the unit hydrograph is 9750 cfs/in. Assume Snyder's synthetic unit hydrograph applies.
(*b*) Determine the two-hour unit hydrograph for the upper 70-mi^2 area of the same watershed, which has $L = 12.6$ mi and $L_c = 7.4$ mi. The values of W_{75} and W_{50} for the entire 100-mi^2-area watershed are 2.0 h and 4.2 h, respectively.

7.7.8 The Gimlet Creek watershed at Sparland, Illinois, has a drainage area of 5.42 mi^2; the length of the main stream is 4.45 mi and the main channel length from the watershed outlet to the point opposite the center of gravity of the watershed is 2.0 mi. Using $C_t = 2.0$ and $C_p = 0.625$, determine the standard synthetic unit hydrograph for this basin. What is the standard duration? Use Snyder's method to determine the 30-minute unit hydrograph for this watershed.

7.7.9 The Odebolt Creek watershed near Arthur, Ohio, has a watershed area of 39.3 mi^2; the length of the main channel is 18.10 mi, and the main channel length from the watershed outlet to the point opposite the centroid of the watershed is 6.0 mi. Using $C_t = 2.0$ and $C_p = 0.625$, determine the standard synthetic unit hydrograph and the two-hour unit hydrograph for this watershed.

7.7.10 An 8-mi^2 watershed has a time of concentration of 1.0 h. Calculate a 10-minute unit hydrograph for this watershed by the SCS triangular unit hydrograph method.

Determine the direct runoff hydrograph for a 20-minute storm having 0.6 in of excess rainfall in the first 10 minutes and 0.4 in in the second 10 minutes.

7.7.11 A triangular synthetic unit hydrograph developed by the Soil Conservation Service method has $q_p = 2900$ cfs/in, $T_p = 50$ min, and $t_r = 10$ min. Compute the direct runoff hydrograph for a 20-minute storm, having 0.66 in rainfall in the first 10 minutes and 1.70 in in the second 10 minutes. The rainfall loss rate is $\phi = 0.6$ in/h throughout the storm.

7.8.1 For the data given in Prob. 7.4.4, use the assumption of constant rainfall intensity in hours 4–6 to construct the S-hydrograph. Use the S-hydrograph to calculate the one-, three-, and six-hour unit hydrographs.

7.8.2 For the data given in Prob. 7.4.5, use the assumption of constant rainfall intensity for six hours to construct the S-hydrograph for this watershed. From the S-hydrograph, determine the 2-, 6-, and 12-hour unit hydrographs for this watershed.

7.8.3 The ordinates of a one-hour unit hydrograph specified at one-hour intervals are (in cfs/in): 45, 60, 22, 8, and 1. Calculate the watershed area, the S-hydrograph and the two-hour unit hydrograph for this watershed.