Towards an Algebra for Terrain-Based Flow Analysis

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Abstract

Topography is an important land surface attribute for hydrology that, in the form of Digital Elevation Models (DEMs), is widely used to derive information for the modeling of hydrologic processes. Much hydrologic terrain analysis is conditioned upon an information model for the topographic representation of downslope flow derived from a DEM, which enriches the information content of digital elevation data. This information model involves procedures for removing spurious sinks, deriving a structured flow field, and calculating derivative surfaces. We present a general method for recursive flow analysis that exploits this information model for calculation of a rich set of flow-based derivative surfaces beyond current weighted flow accumulation approaches commonly available in Geographic Information Systems, through the integration of multiple inputs and a broad class of algebraic rules into the calculation of flow related quantities. This *flow algebra* encompasses single and multi-directional flow fields, various topographic representations, weighted accumulation algorithms, and enables untapped potential for a host of application-specific functions. We illustrate the potential of flow algebra by presenting examples of new functions enabled by this perspective that are useful for hydrologic and environmental modeling. Future opportunities for advancing flow algebra functionality could include the development of a formulaic language that provides efficient implementation and greater access to these methods. There are also opportunities to take advantage of parallel computing for the solution of problems across very large input datasets.

Introduction

The land surface plays a crucial role in the hydrologic cycle by controlling the partitioning of precipitation into various components of runoff, infiltration, storage and evapotranspiration. Topography is arguably the most important land surface attribute for hydrologic applications since it serves to define watersheds, the most basic hydrologic model element. Beven and Kirkby's TOPMODEL (Beven and Kirkby, 1979) has enjoyed widespread success as one of the first hydrologic models to take advantage of digital representations of topography. Terrain analyses based on digital elevation data are increasingly used in hydrology (e.g. Wilson and Gallant, 2000).

Digital representation of topography is usually through one of three data structures (Wilson and Gallant, 2000): (1) regular grids, (2) triangulated irregular networks, and (3) contours (Figure 1). Square grid digital elevation models (DEMs) have emerged as the most widely used data structure, because of their simplicity and ease of computer implementation. Triangulated irregular networks (TINs) have also found widespread use (Jones et al., 1990; Nelson et al., 1999; Tucker et al., 2001) because they can be adapted to the scale or detail of terrain information. The contour-based, stream tube concept first proposed by Onstad and Brakensiek (1968) has also been used in hydrology to avoid the bias associated with grid data structures (O'Loughlin, 1981; 1986; Moore et al., 1988; Moore and Grayson, 1991; Grayson et al., 1992; Dawes and Short, 1994). Despite their potential, contour based methods have not seen widespread application, perhaps due to their complexity, with current implementations requiring careful handling of special cases (Wilson and Gallant, 2000). The specific work reported in this paper relies on grid digital DEMs, although many of the concepts are generic and extend to TIN or contour/flow tube elements.

Since the first grid-based DEMs appeared in the late 1980's there has been rapid ongoing improvement of DEM data available to the hydrologic community, including the U. S. National Elevation Dataset which provides seamless coverage across the United States (at 10 m resolution in many locations). Worldwide, the Shuttle Radar Topography Mission (SRTM) data provides 90 m resolution coverage globally with higher resolution data available in some places. DEM acquisition techniques based on LIDAR (light detection and ranging) are producing centimeter accuracy high-resolution DEM datasets. We stand at a threshold of improvement in surface topography precision due to LIDAR that provides both opportunities and computing challenges. Rapid expansion of digital elevation applications is also driven by increasing power available in personal computers and the capability to rapidly download and process DEM data. This is leading to increased incorporation of terrain derivatives into analysis, in many fields, including hydrology and environmental modeling. This paper contributes to methods for development of flow-related terrain derivatives that might enhance such analyses.

Information science includes the precise representation of physical environments using data models that enhance the capability for analysis and integration of information. This paper examines data models for the representation of flow over terrain in Geographic Information Systems (GIS) and presents new formalism for deriving flow-based information useful for hydrologic and environmental modeling. A basic underlying assumption is that water and its associated constituents move downhill. Terrain-based flow models enrich the information available from a DEM by deriving a structured digital representation of the flow field, which serves as the foundation for calculation of a wide range of flow-related quantities, the most basic of which is contributing area. The algorithm for calculating contributing area can be generalized to include additional information and rules, and to produce additional spatial fields of interest. Here we review methods for calculating terrain-based flow fields and existing algorithms for efficient derivation of flow-based information. Collectively, these methods form the conceptual basis for the encompassing formalism of *flow algebra*. Flow algebra provides a general approach for the incorporation of rules into flow-related calculations that encompass existing flow accumulation methods as special cases while allowing for the development of additional applications. While derived with the basic downhill assumption in mind, flow algebra is not limited solely to the movement of water over terrain. The formalism applies to any non-circulating (non-looping) flow field and flow directions used in flow algebra can be derived from any potential surface. Flow fields derived from the gradient in any potential field (such as topographic slope in a gravitational field) are non-circulating because flow is from high to low potential. Flow algebra concepts thus have broad application for modeling the natural environment.

This paper is organized as follows. We first describe the terrain based flow data model. This is a review of existing work from a data modeling perspective and presents the digital representation of the terrain flow field as the foundation for recursive flow analysis, presented in the next section and flow algebra presented in the section following. The section on recursive flow analysis reviews how the digital representation of the flow field supports the calculation of derivative flow related surfaces. This then leads in to the section on *flow algebra* as a generalization of recursive flow analysis that encompasses the use of algebraic rules in recursive calculations of flow related derivative surfaces. We then present a section with examples that illustrate the capability of flow algebra and conclude with some thoughts on future directions for the development and use of flow algebra in terrain analysis for hydrologic and environmental modeling.

The Terrain-Based Flow Data Model

The terrain-based flow data model comprises a digital representation of terrain (Figure 1) and a representation of the flow field that connects adjacent model elements enabling the routing of flow over a terrain surface and providing the basis for terrain-based flow calculations. This section reviews existing methods for the construction of the terrain-based flow model comprising drainage correction and calculation of the flow field representation for grid DEMs, and some of the hydrologic and environmental modeling work that has exploited this model.

In grid DEMs, sinks comprised of grid cells surrounded by higher-elevation neighbors occur due to deficiencies in DEM production processes and generalization in the representation of terrain (Jenson and Domingue, 1988; Jenson, 1991). Drainage correction that removes sinks is an important, but not essential, first step in the development of the terrain-based flow information model. Drainage correction is the processes of altering (correcting) the DEM to remove these sinks and a DEM that has had all sinks removed is referred to as hydrologically correct. Care needs to be exercised not to "correct" non spurious sinks or alter the DEM surface so much as to introduce further error into hydrologic analyses. The choice as to whether to remove sinks or not, therefore needs to be based upon the physical use and interpretation of the results.

Several efficient implementations of sink filling have been developed (Planchon and Darboux, 2001; Arge et al., 2003). Breaching or carving alterations to the DEM to allow drainage through barriers have also been suggested using either a 3-4 grid cell search (Garbrecht and Martz, 1995; Garbrecht and Martz, 1997) or by tracing downwards from the pour point until an elevation lower than the sink is found then carving a path from the sink to the lower elevation (Soille et al., 2003). Soille (2004) developed a logical integration of the sink filling and carving approaches that minimizes overall modification of the DEM by optimizing between raising the elevation of terrain within sinks and lowering the elevation of terrain along sink outflow paths. This approach provides a hydrologically correct DEM that is as close as possible to the original DEM data. Grimaldi et al. (2007) suggested a physically-based method that employs solutions from a landscape evolution model to remove sinks. A concern with this approach is that it favors the landscape-evolution model over real terrain data, in some case altering the original DEM even more than a filling approach in order to bring the DEM surface into conformity with model solutions.

The most common procedure for routing flow over a terrain surface represented by a grid DEM is the eight-directional method (D8) first proposed by O'Callaghan and Mark (1984). In this model, the direction of steepest descent towards one of the eight (cardinal and diagonal) neighboring grid cells is used to represent the flow field (O'Callaghan and Mark, 1984; Marks et al., 1984; Band, 1986; Jenson and Domingue, 1988; Mark, 1988; Morris and Heerdegen, 1988; Jenson, 1991; Martz and Garbrecht, 1992). In cases where the steepest descent cannot be determined, a broader search radius or random selection from among ties may be used. Garbrecht and Martz (1997) presented a method for the routing of flow across flat surfaces both away from higher terrain and towards lower terrain that improved over prior methods. However, the D8 approach is limited because it can assign flow to only one of eight possible directions, each separated by 45^o in a square grid (Fairfield and Leymarie, 1991; Costa-Cabral and Burges, 1994; Tarboton, 1997).

Multiple flow direction methods (Quinn et al., 1991; Freeman, 1991; Tarboton, 1997; Seibert and McGlynn, 2007) have been suggested as an attempt to solve the limitations of D8. Multiple flow direction methods proportion the outflow from each element between one or more downslope elements. They thus introduce dispersion (spreading out) into the flow with the goal to represent downslope flow in an average sense. A challenge in developing multiple flow direction approaches using grid DEMs involves balancing the introduction of dispersion against bias from routing flow along grid directions. The D-infinity (D∞) multiple flow direction model (Tarboton, 1997) represents flow direction as a vector along the direction of steepest downward slope on eight triangular facets centered at each grid cell. Flow from a grid cell is shared between the two downslope grid cells closest to the vector flow angle based on angle proportioning. Siebert and McGlynn (2007) introduced an extension to D∞ called MD∞ that combines ideas from Tarboton (1997) with Quinn et al. (1991). The MD∞ approach calculates slopes on triangular facets, but then proportions the flow between multiple downslope directions on triangular facets, thereby accounting for divergent situations where flow between more than two downslope grid cells is likely. MD∞ introduces more dispersion than D∞, but reduces some of the grid bias that D∞ creates in divergent situations. Figure 2 illustrates the representation of flow on a plane surface by single and multiple flow direction methods.

All flow field methods assign or proportion flow from each grid cell to one or more of its adjacent neighbors. In grid DEMs the basic model element is a grid cell, but the same concepts can be applied to any set of topologically connected model elements (Figure 3). Grid, TIN, and contour-flow-tube-element flow field assignments are all subject to the general condition that the proportions assigned to each downslope element are positive and should satisfy the conservation constraint:

$$\sum_i P_{ij} = 1 \qquad (1)$$

where P_{ij} is the proportion of flow going from element i to a neighboring element j and the sum is over all the neighboring elements. For the D8 grid model these proportions are either 1 (connected) or 0 (not connected). For the multiple flow direction models these proportions fall between 0 and 1 for each neighboring element. There is also a requirement that flow is non-circulating such that no portion of flow leaving one element ever returns to the same element after passing through one or more of its neighbors.

Many measures useful in hydrologic and environmental modeling have been derived from this flow model. Without being comprehensive, these include the wetness index (Beven and Kirkby, 1979), a quasi-dynamic wetness index (Barling et al., 1994), terrain stability (Montgomery and Dietrich, 1994; Pack et al., 1998a; 1998b; 2001; Borga et al., 2002), erosion (Roering et al., 1999; Jones, 2002; Istanbulluoglu et al., 2002; 2003; Cochrane and Flanagan, 2003), contaminant transport (Ning et al., 2002; Endreny and Wood, 2003), and riparian buffers (Tomer et al., 2003; McGlynn and Seibert, 2003; Baker et al., 2006). Typically these measures have involved combining existing fields (e.g., slope) with outputs of an accumulation operation (e.g., specific contributing area).

Recursive Flow Analysis

Once a flow data model comprising a set of flow proportions for each model element is defined, it may be used to evaluate contributing area and other accumulation derivatives across a DEM domain. In the most general sense, the flow field derived from a DEM defines the surface connectivity between any two parts of a landscape. Given a flow field, the general accumulation function is defined by an integral of a weight or loading field $r(\underline{x})$ over a contributing area, CA.

$$A(\underline{x}) = A[r(\underline{x})] = \int_{CA} r(\underline{x})d\underline{x} \tag{2}$$

In this expression $\underline{x}$ represents the location of an arbitrary point in the domain, $A(\underline{x})$ represents the result of the accumulation function evaluated at that arbitrary point, and A[.] denotes the accumulation operator, which operates on $r(\underline{x})$ to get the result $A(\underline{x})$. Figure 4 illustrates this concept. For a direct contributing area calculation, the weighting field, $r(\underline{x})$, is set equal to 1. In an example calculation of streamflow from excess rainfall, the weighting field would be set equal to rainfall minus infiltration.

Mark (1988) presented a recursive algorithm for evaluation of accumulation in the D8 case that was extended to multiple flow direction methods by Tarboton (1997). Numerically flow accumulation is evaluated recursively for each element as

$$A_i = A(\underline{x}_i) = r(\underline{x}_i)\Delta \; + \sum_{\{k:P_{ki}>0\}} P_{ki}A(\underline{x}_k) \tag{3}$$

where $\underline{x}_i$ is a location in the field represented numerically by a model element such as grid cell in a DEM and $A_i=A(\underline{x}_i)$ represents the accumulation at that element. The model element area is Δ and the notation $\{k:P_{ki}>0\}$ denotes that summation is over the set of k values such that $P_{ki}>0$ (i.e., summing the contribution from neighboring elements k to element i). In other words, accumulated flow at any model element is the sum of flow arising from that element and flow arising from all contributing neighboring elements, each weighted according to the proportion of flow it contributes. This is a recursive definition because the accumulated flow for any model element depends upon the accumulated flow of adjacent upslope elements. Recursive definition includes a requirement that in tracing each path upstream, one must eventually arrive at a source element that has no other elements draining into it. This “termination requirement” is satisfied as long as the flow field is non circular. Contributing area, as we have defined it in (2) and (3) above, is ill-posed for any flow field that includes looping. Appendix A presents pseudocode for the general upslope recursive algorithm for evaluating equation (3) that can be used for any flow field expressed in terms of the proportion of flow between elements, P_{ki}.

Figure 5 illustrates the contributing area computed using D8 flow directions. In this case P_{ki} is either 1 or 0 and is assigned to the neighboring element in the direction of steepest downwards slope. The streaks aligned with grid directions illustrate the grid bias of the D8 approach. Figure 6 illustrates the contributing area computed using D∞ flow directions. In this case the proportions P_{ki} are proportioned among downslope neighbors, thus reducing grid bias and providing a contributing area result that is smoother, due to the dispersion, and appears to be better reflective of the topography indicated by the contour lines. Tarboton (1997) evaluated the differences between D8 and D∞ for theoretical surfaces where the contributing area is known and showed that the D∞ calculations had smaller bias and mean square error.

The recursive algorithm presented above is an upslope recursion because it examines all the elements upslope from the element at which the quantity of interest is being evaluated. Tarboton (2003) presents a number of other functions that exploit upslope recursion for the development of hydrologically useful quantities, such as downslope influence, decaying accumulation, and concentration limited accumulation. *Downslope influence*, illustrated in Figure 7, represents a special case of weighted flow accumulation from any target set of elements $\underline{y}$ within a given domain so that

$$I(\underline{x}|\underline{y})=A[i(\underline{x}|\underline{y})] \qquad (4)$$

where A[.] is the weighted accumulation operator presented in (3). Isolation of the contribution from the target zone $\underline{y}$ is accomplished with the condition that $r(\underline{y})=1$ for $\underline{x} \in \underline{y}$ and $r(\underline{x})=0$ elsewhere, denoted by a (1,0) indicator function $i(\underline{x}|\underline{y})$ on the set $\underline{y}$. $I(\underline{x}|\underline{y})$ is the contribution (influence) from the set of elements $\underline{y}$ at each element $\underline{x}$ in the map. Downslope Influence is useful in hydrology, water quality analysis, and land management for tracking where contaminants or sediment from a specific source are expected to move. Contributions from a set of source elements can follow several different pathways in a multi-direction flow field. The level of influence along these pathways can decrease with transport distance if source contributions are spread across a greater number of receiving elements in a divergent flow field.

Recursive flow analysis can also examine elements downslope from the element at which the quantity of interest is being evaluated. A function that uses this idea (Tarboton, 2003) is the *Upslope Dependence* function, which is the inverse of Downslope Influence. Upslope dependence of a set of model elements $\underline{y}$, may be related to downslope influence by

$$D(\underline{x}\,|\,\underline{y}) = I(\underline{y}\,|\,\underline{x}) \qquad (5)$$

$D(\underline{x}|\underline{y})$ gives the proportion of flow from a model element $\underline{x}$ than contributes to (eventually flows through) one or more of the elements in the set $\underline{y}$. In this case the target $\underline{y}$ is downslope rather than upslope of the elements being evaluated. Evaluation of this function requires reversal of the direction in which the flow direction field is traversed. Whereas the accumulation operator in (3) tracks the proportion of flow from a set of elements k to a receiving element i if $P_{ki}>0$, here the operator moves in the opposite direction, $P_{ik}>0$, such that

$$R_i = R(\underline{x}_i) = R[r(\underline{x}_i)] = r(\underline{x}_i)\Delta + \sum_{\{k:P_{ik}>0\}} P_{ik} R(\underline{x}_k) \qquad (6)$$

In this expression R[.] is a general reverse weighted accumulation operator that operates on the weighting field $r(\underline{x})$ tracking the downslope amount back up the slope. The result at each model element, denoted $R_i=R(\underline{x}_i)$ is the sum of the contribution from element $\underline{x}_i$, $r(\underline{x}_i)\Delta$, and the accumulation from downslope elements, $\underline{x}_k$, according to the proportions P_{ik}. Upslope dependence of the target set $\underline{y}$ is evaluated by setting $R(\underline{x})=1$ for $\underline{x} \in \underline{y}$, initializing these elements as 1, setting $r(\underline{x})=0$ elsewhere and evaluating equation (6) recursively. Pseudocode for recursive downslope, or reverse, flow accumulation that evaluates equation (6) is given in Appendix B.

There is an irony in the terminology here, in that evaluation of upslope dependence requires recursion in the downslope direction. This occurs because evaluation of whether an element is upslope of a target area, requires one to search downslope. Figure 8 illustrates how Upslope Dependence can be used to identify the area comprising elements that contribute some fraction of their area to the flow through a target area. Given this, the upslope dependence function can be useful for tracking the likely origins of sediment or other dissolved contaminant at a receiving location. The upslope dependence function can also be used for delineating the area draining to a watershed outlet. It should be noted that in contrast to single direction contributing areas, multiple flow direction approaches allow a single model element to contribute to both the target set (i.e., $i(\underline{x}|\underline{y}) = 1$) as well as elements outside the target set (i.e., $i(\underline{x}|\underline{y}) = 0$), or to more than one catchment outlet. Thus, to identify discrete watersheds draining to separate outlets, a rule based on the largest upslope dependence value or an upslope dependence threshold is needed.

Towards a Flow Algebra

Examination of the recursive flow analysis examples presented above reveals some generality and pattern to these calculations: (1) multiple direction accumulations rely on weighted flow proportioning whereas single direction accumulations are a special case where all flow follows one pathway; (2) flow proportioning can occur to any number of neighboring model elements, so long as it conforms to the conservation constraint; (3) recursion can occur in both upslope and downslope directions; and (4) accumulations can be weighted by additional field(s) (e.g., rainfall minus infiltration). This capability, at least for upslope recursions, is available in flow accumulation functions in general purpose Geographic Information System software. However, we suggest here that recursive flow analysis need not be limited to the incorporation of additional weight fields into flow accumulation. Rather, what is needed is the ability to involve one or more additional fields in the accumulation functions that operate during the recursion according to a set of logical rules. We call these general rules for flow related calculations *flow algebra*. Because these general rules encompass all existing flow-related procedures, what we present here comprises a unifying approach for understanding past, present, and future flow related calculations. By exposing the generality of flow-field related calculations, we hope to suggest a direction for software development that will enable and stimulate generation of additional flow-derived measures useful in hydrology and environmental modeling.

Flow algebra logic exploits the recursive evaluation methodology illustrated in equation (3). Recursion serves to simplify the evaluation of a flow algebra function from its global or zonal integral definition, such as in equation (2), to a local evaluation where the function value at an element depends only on variables at that element and at *either* elements immediately upstream or downstream in the flow network, but not both at the same time. Flow algebra also generalizes the capability of zonal integral functions, enabling the evaluation of quantities that could not be defined in terms of a zonal integral because the result depends on both the flow field as well as local rules or additional value fields. We distinguish within flow algebra between simple input variables and variables with recursive dependence. Simple input variables or fields, denoted $\underline{\gamma}(\underline{x})$, are fully quantified before the evaluation of a flow algebra expression. Variables that have recursive dependence on the flow field, denoted $\underline{\theta}(\underline{x})$, are quantified during the course of evaluating a flow algebra expression.

In general, a flow algebra expression may be written as

$$\underline{\theta}(\underline{x}_i) = f(\underline{\gamma}(\underline{x}_i), \underline{P}_{ki}, \underline{\theta}(\underline{x}_k), \underline{\gamma}(\underline{x}_k)) \quad (7)$$

for an upstream function, or

$$\underline{\theta}(\underline{x}_i) = f(\underline{\gamma}(\underline{x}_i), \underline{P}_{ik}, \underline{\theta}(\underline{x}_k), \underline{\gamma}(\underline{x}_k)) \quad (8)$$

for a downstream function. The function f(.) may include any mathematical operators such as: +, -, ÷, ×, summation, conditional, logical, trigonometric and mathematical functions. In this expression $\underline{\theta}(\underline{x}_i)$ is a list (of dimension m) of the recursive variables being evaluated at location i by the expression. $\underline{\gamma}(\underline{x}_i)$ is a list (of dimension q) of all simple input variables. $\underline{P}_{ik}$ or $\underline{P}_{ki}$ is a vector giving the proportion of flow from the first subscript element to the second subscript element, defined over all k for which $P_{(..)}$ is non-zero. $\underline{P}_{ik}$ or $\underline{P}_{ki}$ is of dimension n where n represents the number of connected neighbor nodes. $\underline{\theta}(\underline{x}_k)$ is a list of all recursive variables evaluated at each neighbor location k. It has dimension m×n. $\underline{\gamma}(\underline{x}_k)$ is a list of simple input variables at each neighbor node k. It has dimension q×n.

The recursive variables, $\underline{\theta}(\underline{x})$, appear on the right-hand side of the expression because evaluation of the expression at location $\underline{x}$ depends on the values for these variables at adjacent logical network nodes, either upstream or downstream. With this structure, not only can $\underline{\gamma}(\underline{x})$ be applied as a weight, both $\underline{\gamma}(\underline{x})$ and $\underline{\theta}(\underline{x})$ fields can be applied during the calculation of any quantity with recursive dependence. A flow algebra expression is either of type "upstream" (e.g., contributing area, downslope influence) or "downstream" (e.g., upslope dependence, reverse accumulation) depending on whether the functional dependence is on upstream or downstream quantities. Recursive dependence upon both upstream and downstream variability in the same expression is not allowed because such recursions would not terminate. Appendix C gives general pseudocode for the implementation of an upstream flow algebra function. The similarity of this to flow accumulation (Appendix A) is apparent. Downstream flow algebra is obtained by reversing P_{ki} to P_{ik}. Upstream and downstream flow algebra is similar in all other respects.

Flow algebra expands upon the concept of map algebra available in popular GIS systems by the inclusion of flow field operations. Map algebra involves point-by-point (cell-by-cell) mathematical operations between spatial fields. Flow algebra adds to this capability by incorporating operations based on the flow field and algebraic or functional descriptions of how the quantity being modeled is related to, and involved with, the flow field.

Because flow algebra encompasses multidirectional flow algorithms, it is applicable to any numerical representation of a flow field, including single or multiple flow direction grids, Voronoi polygons based upon a TIN discretization, or flow net model elements based upon a contour and flow line discretization. Each flow field representation has an underlying logical network structure defining the connectivity between elements (Figure 3). This may be implicit (as in the case of grids) or explicit (for Voronoi polygons and flow net model elements). Flow algebra elements could also be topographically delineated catchments. For example, the Arc Hydro data model (Maidment, 2002) provides connectivity between stream reaches and stream reach catchments (the area draining directly to a stream reach) within a stream network and implements accumulation functions using reach catchments as model elements.

Examples of functions constructed using Flow Algebra

This section gives examples to that illustrate how flow algebra may be used to extend the functional capability of recursive flow analysis through the incorporation of rules into the recursive evaluation methodology. The examples have an increasing level of complexity so as to develop basic concepts using simple functions and then by gradually adding modifications, illustrate potential for more specific applications.

A natural measurement derived from any flow field is that of *distance along a flow pathway*. Specifically, we consider here the distance in a downslope direction from each model element to a target set, such as a stream or catchment outlet though upslope distances may also be defined using an upslope recursion. In hydrologic analyses, flow lengths have been used to characterize geomorphologic instantaneous unit hydrographs (Rodriguez-Iturbe and Valdes, 1979) estimate water residence times (McGuire et al., 2005), contrast geomorphologic versus hydrodynamic attenuation/dispersion (White et al., 2004), and characterize water quality (Alexander et al., 2000; Soranno et al., 1996). A variety of ecological analyses have used flow path distances to understand the influence of the spatial arrangement of watershed attributes on water quality and biotic responses (King et al., 2004; Frimpong et al., 2005; King et al., 2005; Van Sickle and Johnson, 2008).

In the D8 model, flow can only proceed to a single downslope element. D8 flow length calculations are consequently relatively straight-forward and comprise accumulation of cardinal (Δx, Δy) or diagonal ($\sqrt{\Delta x^2 + \Delta y^2}$) cell traverses, where Δx and Δy are element dimensions. In a multiple direction flow model, the distance from any model element $\underline{x}_i$ to another element $\underline{x}_j$ is not uniquely defined. Flow that originates at element $\underline{x}_i$ may arrive at $\underline{x}_j$ by a number of distinct pathways and flow length is thus defined by a distribution rather than a single number. Bogaart and Troch (2006) proposed calculating the average of this length distribution by weighting by the fraction of flow directed along a particular flow pathway. We present a general implementation below using flow algebra. Practically speaking, the full length distribution cannot be accumulated easily over large domains due to excessive computational demands, however distance functions that retain the longest and shortest paths may also be defined.

For the evaluation of average distance using flow algebra, the vector of simple inputs, $\underline{\gamma}(\underline{x})$, is comprised of the coordinates of the center of each element and a target set indicator $\underline{y}$ (e.g. $\underline{y}_i$=1 on the stream and 0 off the stream). The vector of recursive variables, $\underline{\theta}(\underline{x})$, comprises the average distance to the target set from element $\underline{x}_i$, denoted $ad(\underline{x}_i)$. Average distance is calculated using a downslope recursion with flow algebra expression f(.), equation (8), defined as:

if $y_i=1$ (if on the indicator set)

$ad(\underline{x}_i)=0$

else

$$ad(\underline{x}_i) = \sum_{\{k:P_{ik}>0\ \&\ ad(\underline{x}_k)\geq 0\}} P_{ik}(dist(\underline{x}_i,\underline{x}_k)+ad(\underline{x}_k)) \Big/ \sum_{\{k:P_{ik}>0\ \&\ ad(\underline{x}_k)\geq 0\}} P_{ik} \tag{9}$$

The extra condition $ad(\underline{x}_k)\geq 0$ is placed in the summation to accumulate only those elements for which the average distance is defined, because distance is not defined for those elements with no downslope elements in the target set. Division by the sum of proportions is to account for partial contribution of a model element to downslope elements for which distance is defined. In most case, the denominator will be equal to one except, for example, when a downslope element flows into a neighboring catchment and out of the domain in which case $ad(\underline{x}_k)$ will be undefined. The function $dist(\underline{x}_i, \underline{x}_k)$ evaluates the geometric distance between the center of elements i and k.

Similarly, the longest distance to the target set from each element $\underline{x}_i$, denoted $ld(\underline{x}_i)$, is calculated using a downslope recursion

if $y_i=1$

$ld(\underline{x}_i)=0$

else

$$ld(\underline{x}_i) = \max_{\{k:P_{ik}>0\ \&\ ld(\underline{x}_k)\geq 0\}} (dist(\underline{x}_i,\underline{x}_k)+ld(\underline{x}_k)) \tag{10}$$

where for each downslope neighbor ($P_{ik} > 0$) the function selects the maximum of the longest distance from that neighbor plus the distance to that neighbor. The shortest distance, $sd(\underline{x}_i)$, is calculated as:

if $y_i=1$

$sd(\underline{x}_i)=0$

else

$$sd(\underline{x}_i) = \min_{\{k:P_{ik}>0\ \&\ sd(\underline{x}_k)\geq 0\}} (dist(\underline{x}_i,\underline{x}_k)+sd(\underline{x}_k)) \tag{11}$$

It may be of practical interest to weight the flow distance to calculate distance differently across a set of element values. A *weighted flow distance* may be calculated by adding a weight field, $w(\underline{x}_i)$, to the input vector $\underline{\gamma}(\underline{x})$. A flow algebra expressions for weighted flow distance, similar to equation (9) above is:

$$ad(\underline{x}_i) = \sum_{k:P_{ik}>0} P_{ik}\left(\frac{w(\underline{x}_i)+w(\underline{x}_k)}{2}dist(\underline{x}_i,\underline{x}_k)+ad(\underline{x}_k)\right) \Bigg/ \sum_{\{k:P_{ik}>0\ \&\ ad(\underline{x}_k)\geq 0\}} P_{ik} \tag{12}$$

The weights associated with the originating and receiving elements are averaged and multiplied by the distance between elements in this calculation.

Weighted distances have recently been applied to the problem of scaling filtering effects of streamside forests and wetlands, which have been observed to reduce concentrations of dissolved nutrients along field-to-stream transects. Baker et al., (2006) used distances measured from row crop agriculture to streams weighted by the presence of forest or wetlands along each flow pathway to characterize the extent of riparian filtering across catchments. In this calculation, croplands are identified from a land cover raster as potential nutrient sources, whereas potential sinks (buffers) include forest and wetlands along flowpaths between each crop element and the stream (Figure 9A&B). Importantly, forests and wetlands occuring adjacent to the stream but not downslope of a nutrient source *are not considered* in the analysis because they are assumed not to be involved in nutrient transport or filtering. A similar approach was recently used to understand how stream map resolution or seasonal expansion and contraction of stream networks might influence estimation of source-sink connectivity and relative nutrient uptake in streamside forests versus headwater streams (Baker et al., 2007). Figure 9 also illustrates how flow length and connectivity estimates may be altered through the use of single (C) versus multidirectional (D) flow fields. In some cases, alternate pathways identifed by a multidirectional flow field may be less (e.g., label 1 in Figure 9D) or more (e.g., 2 in Figure 9D) buffered compared to single-direction paths. In every case where multidirectional flow dispersion occurs (e.g., 3 Figure 9D), estimates of the area of potential buffer used in buffering will be necessarily greater than when using single directional estimates.

A simple extension of the above recursion, the *Drop Function* is defined for any model element as the elevation difference from a location $\underline{x}_i$ on the land surface to a target region downslope, usually the stream or catchment outlet. In this case, a DEM serves as an additional input field, $\underline{\gamma}(\underline{x})$, providing the value z. McGuire et al. (2005) used MD∞ to accumulate flow in their study of water residence time, but were limited to using D8 for flow distance, flow gradient (drop/distance), and gradient-to-distance ratios. Below we present a flow algebra solution to this problem. Given a multiple flow direction field with flow out of each element being proportioned between downslope model elements, there is no single pathway by which flow from any $\underline{x}_i$ reaches a set of downslope elements $\underline{y}$. The Drop Function may therefore be defined in term of the maximum drop

$$\max_{\{j:Q_{ij}>0\}} (z(\underline{x}_i) - (z(\underline{y}_j)) \tag{13}$$

the minimum drop

$$\min_{\{j:Q_{ij}>0\}} (z(\underline{x}_i) - (z(\underline{y}_j)) \tag{14}$$

or the average drop

$$z(\underline{x}_i) - \sum_{\{k:Q_{ij}>0\}} Q_{ij} \; z(\underline{y}_j) \tag{15}$$

As in flow distance calculations, the target region to which drop is being measured is indicated by the set of elements $\underline{y}$. These may be quite a long way from the element $\underline{x}_i$. A subset of these receive flow from the element $\underline{x}_i$. Q_{ij} denotes the proportion of flow from element $\underline{x}_i$ that eventually gets to $\underline{y}_j$ in the set $\underline{y}$. The maximum drop formula evaluates the elevation drop to the lowest point where flow from element $\underline{x}_i$ enters $\underline{y}$. The minimum drop formula evaluates the elevation drop to the highest location where any flow from element $\underline{x}_i$ enters $\underline{y}$. The average drop formula weights the drop based on the proportion of flow entering element $\underline{y}_j$ at each location. Numerically, these equations are evaluated using a downslope recursion based on the multiple flow proportions P_{ik} giving flow from grid cell i to grid cell k. The maximum drop is calculated as

$$\text{mxdrp}(\underline{x}_i) = \underset{\{k:P_{ik}>0\}}{\text{Max}}(z_i - z_k + \text{mxdrp}(\underline{x}_k)) \tag{16}$$

This adds the drop from i to neighbor k to the longest drop from neighboring element k. The maximum is over all the neighbors that receive a positive proportion of the flow, $P_{ik}>0$. The minimum drop is similarly calculated as

$$\text{mndrp}(\underline{x}_i) = \underset{\{k:P_{ik}>0\}}{\text{Min}}(z_i - z_k + \text{mndrp}(\underline{x}_k)) \tag{17}$$

and the average drop as

$$\text{avdrp}(\underline{x}_i) = \sum_{\{k:P_{ik}>0\}} P_{ik}\ (z_i - z_k + \text{avdrp}(\underline{x}_k)) / \sum_{\{k:P_{ik}>0\ \&\ \text{avdrp}(\underline{x}_k)\geq 0\}} P_{ik} \tag{18}$$

These recursive definitions have the escape condition that mxdrp($\underline{x}_k$), mndrp($\underline{x}_k$) and avdrp($\underline{x}_k$) are 0 for model elements $\underline{x}_k$ that belong to the set of target elements $\underline{y}$.

Similarly, minimum, maximum and average *Rise to Ridge Functions* (rtr) from any element $\underline{x}_i$ may be defined, essentially just by switching i and k in equations (16) to (18) to switch from downslope to upslipe recursion, and renaming the functions

$$\text{minrtr}(\underline{x}_i) = \underset{\{k:P_{ki}\geq T\}}{\text{Min}}(z_k - z_i + \text{minrtr}(\underline{x}_k)) \ \text{ if } \sum P_{ki} > 0,\ 0 \text{ Otherwise} \tag{19}$$

$$\text{maxrtr}(\underline{x}_i) = \underset{\{k:P_{ki}\geq T\}}{\text{Max}}(z_k - z_i + \text{maxrtr}(\underline{x}_k)) \ \text{ if } \sum P_{ki} > 0,\ 0 \text{ Otherwise} \tag{20}$$

$$\text{artr}(\underline{x}_i) = \frac{\sum_{\{k:P_{ki}\geq T\}} P_{ki}\ (z_k - z_i + \text{artr}(\underline{x}_k))}{\sum_{\{k:P_{ki}\geq T\}} P_{ki}} \ \text{ if } \sum P_{ki} > 0,\ 0 \text{ Otherwise} \tag{21}$$

In the rise to ridge functions the escape condition for the recursion is $\sum P_{ki} > 0$ that defines ridge elements as elements that do not have any upslope elements. In equations (19) to (21) we also introduced the option for a user input threshold, T, to control upslope paths from neighbors k that enter element $\underline{x}_i$ that are considered to be upslope.

Transport Limited Accumulation is a flow algebra function that introduces further rules into flow-related calculations. This function is designed to calculate the transport of sediment that may be limited by *both* the sediment supply and the capacity of the flow field to transport sediment. Importantly, this is an example of an algorithm not currently available to general GIS users without the functionality of flow algebra. We have framed the calculation in a general way with supply and transport capacity fields as inputs (components of $\underline{\gamma}(\underline{x})$), so as to apply to any transport process where there is both distributed supply of a substance and a limited capacity for transport of that substance. This function accumulates substance flux subject to the rule that transport out of any model element is the minimum between supply and transport capacity. The total supply is calculated as the sum of transport in to the element from upslope elements plus the supply contribution from the element. This is again a recursive definition, since it depends upon the transport flux from upslope elements. Specifically,

$$T(\underline{x}_i) = \text{Min}(C(\underline{x}_i),\ \sum_{\{k:P_{ki}>0\}} P_{ki} T(\underline{x}_k) + S(\underline{x}_i)) \tag{22}$$

where $C(\underline{x}_i)$ is the transport capacity associated with model element $\underline{x}_i$, and $S(\underline{x}_i)$ the supply (e.g., erosion potential) at model element $\underline{x}_i$, and $T(\underline{x}_i)$ gives the resulting transport limited accumulation flux. If $C(\underline{x}_i)$ exceeds transport to the element plus local supply, then the flux is supply limited and the second term in the Min is chosen. If the available substance from the sum of influx plus local supply exceeds $C(\underline{x}_i)$, then the flux is transport limited and the outflux is the transport capacity, $C(\underline{x}_i)$. Both transport capacity and local supply fields ($C(\underline{x}_i)$ and $S(\underline{x}_i)$) are inputs and thus components of $\underline{\gamma}(\underline{x})$, while the resultant transport limited accumulation flux is the result of recursion on the flow field and thus an element of the vector $\underline{\theta}(\underline{x})$. Another part of $\underline{\theta}(\underline{x})$ and byproduct of this calculation is the deposition $D(\underline{x}_i)$ at any point, calculated as total supply minus actual transport,

$$D(\underline{x}_i) = \sum_{\{k:P_{ki}>0\}} P_{ki} T(\underline{x}_k) + S(\underline{x}_i) - T(\underline{x}_i) \quad (23)$$

$D(\underline{x}_i)$ is 0 at supply limited elements, while at transport limited elements it quantifies the excess of total supply over transport capacity. Comparison of $D(\underline{x}_i)$ to $S(\underline{x}_i)$ is required to distinguish deposition of substance from local supply, versus substance that is transported into an element from another upslope element. This model for accumulation of a substance subject to supply and transport capacity limits is consistent with sediment transport and erosion theory involving the separate processes of detachment and transport (Hairsine and Rose, 1992a; 1992b). Figure 10 illustrates Transport Limited Accumulation. The supply field may be based on erodibility from soil surveys, while transport capacity in this example is based on slope-area relationships (Dietrich et al., 1992; Montgomery and Dietrich, 1994). Reductions in sediment delivery ratios as drainage area increases are naturally modeled by this function due to the trapping of sediment at locations where transport capacity is limited.

Calculation of an *Avalanche Runout Zone* provides another, more comprehensive opportunity to illustrate the generality and potential of flow algebra for calculations involving multiple terrain and flow fields. In this application, avalanche source zones, identified manually using expert knowledge and visual interpretation of maps, is used as input (although there is clearly potential for modeling avalanche source zones based upon topographic attributes such as has been done for landslides, Pack et al., 1998a; 1998b; Tarolli and Tarboton, 2006). The rule for identifying runout zones is that all locations downslope from a source zone are potentially affected up until the energy from the avalanche is depleted. This depletion point is estimated when the slope between the source and the affected area is less than a threshold angle (alpha). The alpha angle is calculated using the distance from the highest point in the source zone to points within the potential runout zone (Figure 11). Distance may be measured either along a straight line or along a flow path. This alpha-angle model is a simple model for avalanche or debris flow runout that is used in practice to evaluate potential hazards (e.g. Schaerer, 1981; McClung and Schaerer, 1993; Iverson, 1997; Toyos et al., 2007). Because evaluation of the runout zone requires looking upslope, flow algebra with upslope recursion is used.

For the avalanche application using a multidirectional flow field, it may be desirable to exclude model elements from the runout zone that receive only a small fraction of flow from the avalanche source. We therefore specify a threshold, T, supplied by the user, that must be exceeded before an element is counted as contributing to a downslope neighbor for the purposes of defining the avalanche runout zone and calculating alpha angle (e.g., $P_{ki} > T$ where T=0.2). T may be input as 0 if all fractional contributions to a downslope element, no matter how small, are to be counted. The avalanche source zone is input as an indicator set as (as_i=1 in avalanche source zone and 0 otherwise). The simple and recursive variables involved in avalanche runout calculation cast in terms of the general flow algebra construct are listed in Table 1.

Table 1. Variables in avalanche runout flow algebra function

Symbol	Description
Simple input variables: $\underline{\gamma}(\underline{x})$	
T	Flow proportion threshold
α	Alpha angle
as	Avalanche source set
x_i, y_i	Coordinates of the center of each element
z_i	Elevation of the center of each element.
Recursive variables: $\underline{\theta}(\underline{x})$	
rz	A runout zone indicator with value 0 to indicate that this grid cell is not in the runout zone and value > 0 to indicate that this grid cell is in the runout zone. Since there may be information in the angle to the associated source site, this variable will be assigned the angle to the source site, denoted as β here (in degrees).
xm, ym	X and Y locations of the source site that has the highest angle to the point in question.
zm	Elevation of the source site that has the highest angle to the point in question
dm	Flow distance from the source site that has the highest angle to the point in question. This is included to allow evaluation of source angles using either straight-line or flow path distances.

The flow algebra expression $f(\underline{\gamma}(\underline{x}_i), \underline{P}_{ki}, \underline{\theta}(\underline{x}_k), \underline{\gamma}(\underline{x}_k))$ for $\underline{\theta}(\underline{x}_i)$ at element $\underline{x}_i$ is evaluated by the pseudocode in Appendix D. Clearly, the suite of inputs and calculated fields in this function far exceeds the capacity of currently available accumulation operators, but is relatively straight forward within the flow algebra construct. Figure 12 illustrates the avalanche runout from three potential source zones computed using $\alpha = 22^o$ for a snow avalanche prone area in Logan Canyon, Utah, USA.

Future Directions

The example flow algebra functions presented above have been programmed for use with grid DEM data using the D∞ multiple flow direction model and included as part of the Terrain Analysis Using Digital Elevation Models (TauDEM) software distributed by the first author (http://www.engineering.usu.edu/dtarb/taudem). Code that implements the recursion is in a C++ library that has been wrapped with a Visual Basic graphical user interface callable from the ESRI ArcGIS geographic information system as an ArcMap toolbar or geoprocessing toolbox, as well as from the open source Mapwindow geographic information system (http://www.mapwindow.org). Source code and compiled executables for a PC are distributed using an open source license. However, such implementations, though based on flow algebra concepts, do not provide the full capability we envision. The recursive algorithms, though compact in terms of coding and efficient in terms of model element evaluations (each element is visited only once), can be inefficient in terms of memory requirements (because at each recursion step the function state is saved on a stack), and are not implemented to take advantage of parallel processing. Broad-scale application of these methods to large datasets will require work to address these limitations. Another step in the implementation process will involve the development of text parsing software for translating user inputs into process-specific recursive accumulations. This software would provide an interface that enables users to design their own combinations of $\underline{\gamma}(\underline{x})$ and $\underline{\theta}(\underline{x})$, specifying their own algebraic and logical rules for custom flow algebra functions.

Despite rapid advances in computer technology, there remains a considerable gap among digital representations of terrain, flow fields and real world observations. As a result, geographic and hydrologic models lag behind current hydrologic theory in their representation of physical processes. Computational modeling frameworks are required that enable the implementation and rapid evaluation of new theories and field based concepts. Flow algebra provides a formalism for thinking about and modeling spatial processes that are related to, or occur embedded within, a flow field. We hope that flow algebra therefore serves to fill some of this gap through the terrain-based flow analyses it enables.

This paper has framed an existing information model for the analysis of flow over terrain in Geographic Information Systems. This model establishes a flow field through (1) drainage correction involving the removal of sinks followed by (2) definition of the flow field through a general multidirectional proportioning of flow from each element among downslope neighbors. The flow field is required to be non-circulating, and as such is suitable for representation of flow derived from the gradient of any potential field. Flow proportions arising from any model element should sum to one to ensure conservation. Once this flow field is defined, a broad class of upstream and downstream recursive functions may be constructed using the formalism of *flow algebra*. We have presented some examples for exploiting this capability including new techniques for addressing the measurement of flow distances, elevation drops, sediment transport, and avalanche run-outs. The new techniques have already been utilized in several distinct applications and they serve to illustrate a small portion of the as yet untapped potential of the recursive flow algebra approach. Although the examples we present, have been developed using grid data structures, the logic of flow algebra is applicable for any set of logically connected elements defining flow in a non-circulating flow field. Many advances in hydrologic modeling have not made their way to GIS applications for the simple reason that they did not work well within a grid data structure, or suffered from limitations due to single flow direction approaches. Advances have also been hampered by the difficulty associated with implementation of rules and logic within flow field related calculations. It is our hope that flow algebra will provide a more inclusive modeling framework for moving across data structures in hydrologic modeling of the natural environment.

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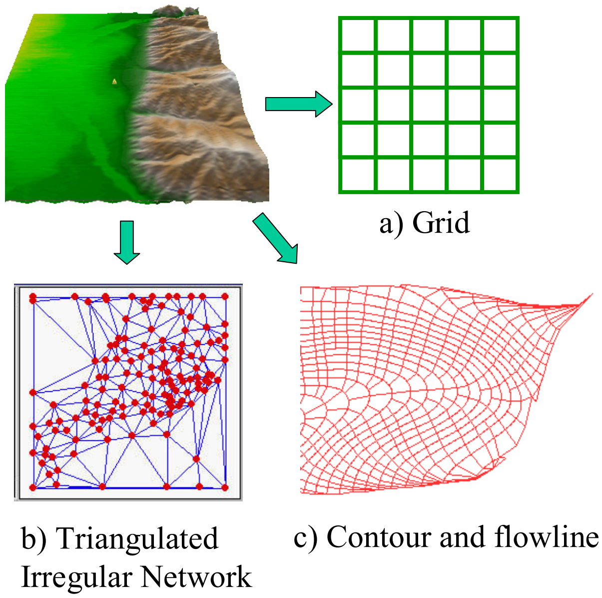

Figure 1. Models for the digital representation of terrain

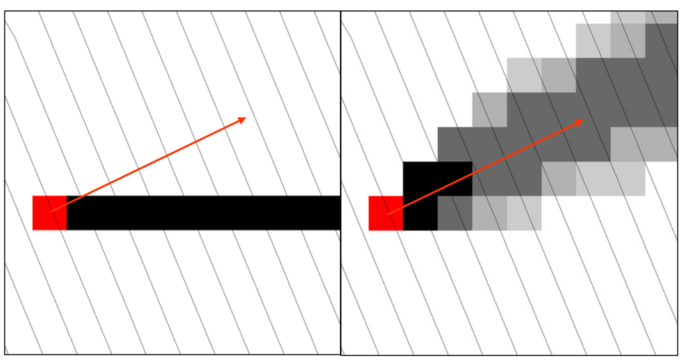

Figure 2. Flow across a plane surface represented by (a) Single flow direction approach, (b) Multiple flow direction approach

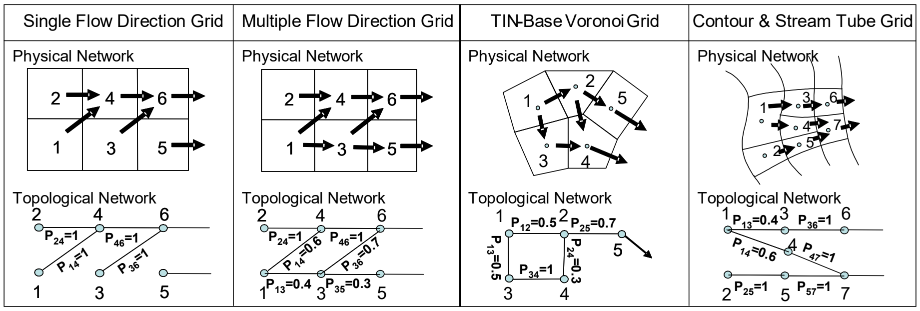

Figure 3. Downslope flow apportioning among topologically connected model elements using different flow field assignments and terrain representations.

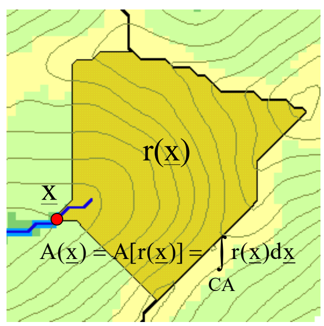

Figure 4. Physical definition of general flow accumulation function

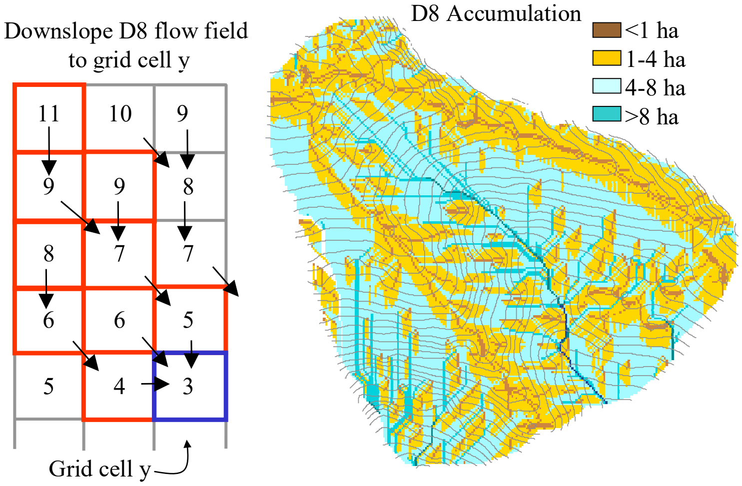

Figure 5. Flow field and contributing area from the D8 method.

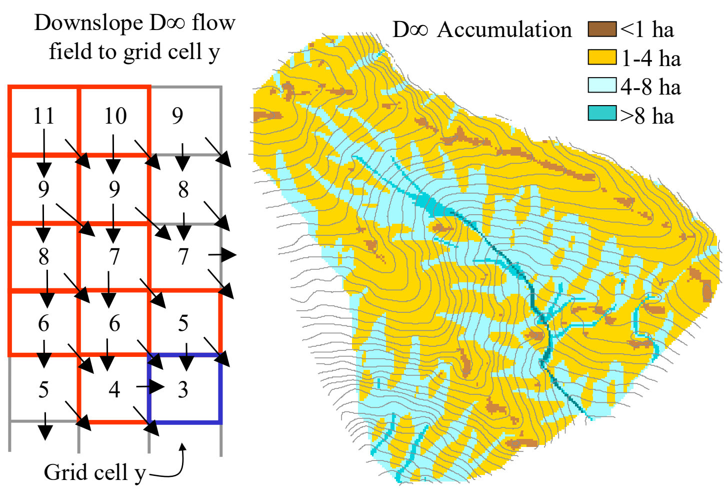

Figure 6. Flow field and contributing area from D∞ method

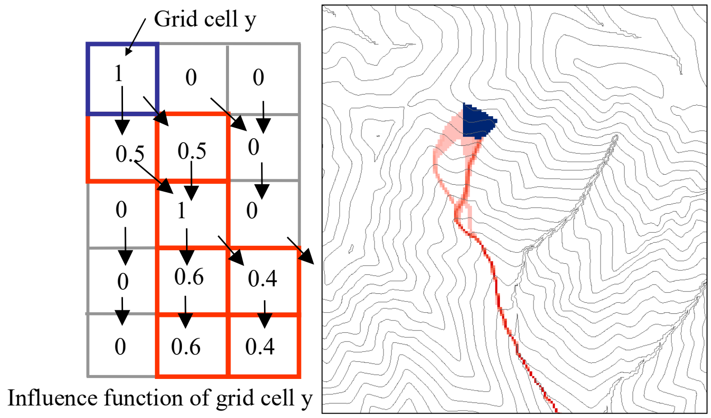

Figure 7. Downslope Influence, calculated as the weighted accumulation from a target set (blue).

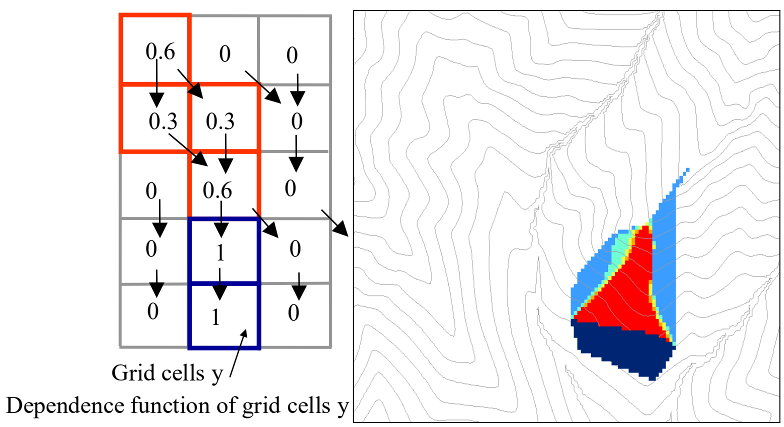

Figure 8. Upslope Dependence quantifies the proportion of flow in a domain (red to light blue) that contribute to a target set (blue). Note cells with fractional contributions along the margin.

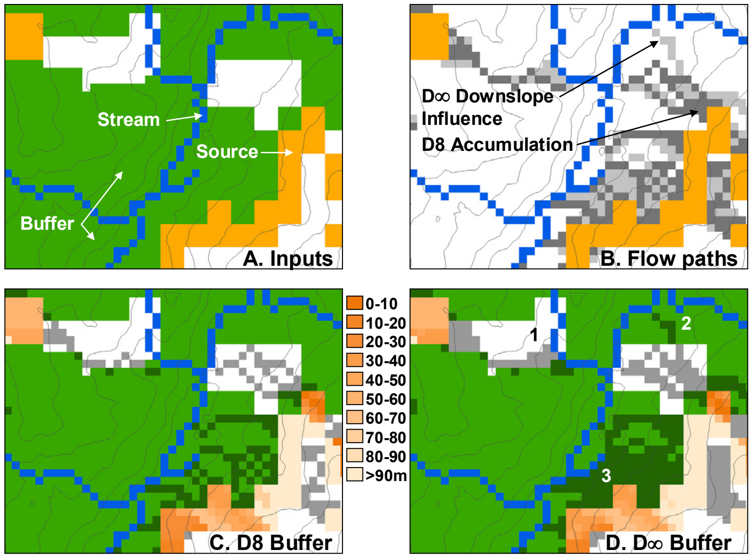

Figure 9. Weighted flow length-to-stream measures used in buffer analyses for water quality modeling in tributaries of Chesapeake Bay, Maryland, USA.

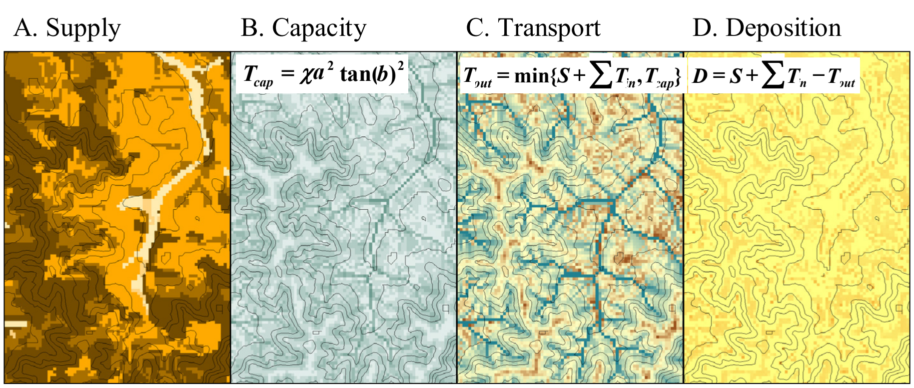

Figure 10. Transport limited accumulation is a function of distributed supply and transport capacity.

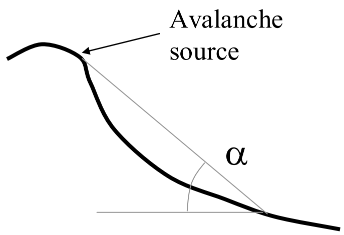

Figure 11. Alpha (α) angle from point in avalanche runout to avalanche source.

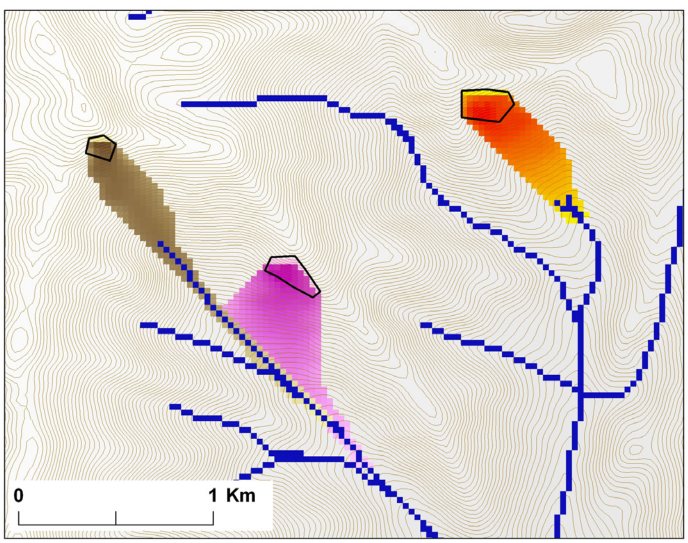

Figure 12. Avalanche Runout zones for Wood Camp Hollow in Logan Canyon, Utah, USA, computed using α = 22o. Contour interval is 10 m. The intensity of the color is scaled by the angle to source, β, subject to the constraint β> α.

APPENDIX A. Pseudocode for Recursive Upslope Flow Accumulation

```
Global variables A_i, r(x_i), P_ij, Δ
Function FlowAccumulation(x_i)
    if A_i is known
    then
            no action
    else
            for each neighbor location x_k indexed by k
                    if(P_ki > 0)then
                            call FlowAccumulation(x_k)
                            //This is the recursive call to calculate area for the neighbor
            Next k
            //  At this point all the neighboring A_k inputs are available
```

$$A_i = r(\underline{x}_i)\Delta \; + \sum_{\{k:P_{ki}>0\}} P_{ki} A_k$$

```
return
```

APPENDIX B. Pseudocode for Recursive Downslope or Reverse Flow Accumulation

```
Global variables R_i, r(x_i), P_ij, Δ
Function ReverseAccumulation(x_i)
    if R_i is known
    then
            no action
    else
            for each neighbor location x_k indexed by k
                    if(P_ik > 0)then
                            call ReverseAccumulation(x_k)
                            //This is the recursive call to the downslope neighbor
            Next k
            //  At this point all the neighboring R_k inputs are available
```

$$R_i = r(\underline{x}_i)\Delta \; + \sum_{\{k:P_{ik}>0\}} P_{ik} R_k$$

```
return
```

APPENDIX C. General Pseudocode for Upstream Flow Algebra Evaluation

```
Global variables γ, θ, P_ij
Function FlowAlgebraUpstream(x_i)
    if θ(x_i) is known
    then
            no action
    else
            for each neighbor location x_k indexed by k
                    if(P_ki > 0)then
                            call FlowAlgebraUpstream(x_k)
```

```
                //This is the recursive call to traverse to an upslope neighbor
        Next k
        //  At this point all the necessary inputs are available
        Evaluate Algebraic expression θ(x_i) = f(γ(x_i), P_ki, θ(x_k), γ(x_k))
    return
```

APPENDIX D. General Pseudocode for Avalanche Runout Zone Evaluation

```
Global variables γ, θ, P_ij
Function AvalancheRunout(x_i)
    if as_i > 0  (if in source zone)
        rz_i = α
        xm = x_i
        ym = y_i
        zm = z_i
        dm = 0
    else
        initialize rz_i = nodata
        For each k with P_ki > T
            if rz_k >= α (neighbor k is in the runout zone)
                if path distance
                    d=dm_k + dist(x_i, y_i, x_k, y_k)    (This is the total distance along flow paths
                        through a neighbor k to the element i)
                else
                    d=dist(x_i, y_i, xm_k, ym_k)    (This is the horizontal distance from element i to
                        the element with maximum angle on the upslope flow path ending at
                        neighbor k)
                zd=zm_k-z_i    (This is the elevation difference from the source on a path coming
                    through neighbor k to cell i)
                β=atan(zd/d)*180/π    (This is the angle in degrees from a source on a path
                    coming through neighbor k to cell i)
                if β ≥ α and β > rz_i    (The set of assignments below assign the vector θ(x_i)
                    using the flow path from a neighbor k for which the angle to the source on
                    that flow path, β, is a maximum)
                    rz_i=β
                    xm=xm_k
                    ym=ym_k
                    zm=zm_k
                    dm=d
    Next k
Return
```