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Scaling and Elevation in River Networks

DAVID G. TARBOTON, RAFAEL L. BRAS, AND IGNACIO RODRIGUEZ-ITURBE

Ralph M. Parsons Laboratory, Massachusetts Institute of Technology, Cambridge

The analysis of large river networks obtained from digital elevation models has given insight into the variation of channel slope with scale. Investigators have recently suggested that channel slopes are self-similar with magnitude or area as a scaling parameter. Our data indicates otherwise; in particular, the variance of channel slope is larger than that predicted by simple self-similarity. This suggests multiscaling. The scaling exponent for the standard deviation is approximately half the corresponding exponent in the relationship of the slope mean to magnitude or area. A model for channel slopes based on a point process of elevation drops along the channel is presented. The model reproduces observed multiscaling properties when the density of elevation increments is related to area (or magnitude) as $A^{-\theta}$.

1. INTRODUCTION

A necessary step in understanding the link between erosive energy balance and network form is the description of the variation of slopes and elevation drops within a river network. This work focuses on the slopes of individual links within channel networks and their variability. Link slope is regarded as a random variable. The issue is how the probability distribution of link slope scales with link magnitude, or area.

The recent paper by *Gupta and Waymire* [1989] suggested a model of link drops with self-similar probability distribution functions (pdf), with area or magnitude taking the role of scaling parameter. In this work we present results based on large digital elevation model (DEM) data sets that show that while correct for the mean their model does not fit the data for higher moments. A more general multiscaling model is needed to characterize the scaling of the full probability distribution of link slope.

The next section describes the self-similar link drop model suggested by *Gupta and Waymire* [1989]. Section 3 then gives the data we have on the scaling of link slopes and drops. In section 4 we present a model for link slopes that focuses on the interplay between size and density of individual elevation increments, idealized as steps. This model fits the observation when the density of elevation increments changes with area as $A^{-\theta}$. This idea provides a mathematical characterization of the multiscaling variability of link slopes. The physical causes and mechanisms resulting in this variability are still an open question. Section 5 summarizes our findings and conclusions. Appendix A gives details on how we extract and analyze channel networks from digital elevation models. Appendix B provides a review of ideas leading up to the current scaling models. It shows how the concepts of order and magnitude as measures of network scale are related. It also relates *Horton*'s [1945] slope law and slope scaling with magnitude as index. Readers not familiar with these issues, or the terminology used, should read Appendexes A and B before proceeding further. Appendix C derives the limit distribution for link slopes when the density of increments becomes infinite while holding the mean and variance of the link slope distribution constant.

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2. SELF-SIMILAR DROP MODEL

Gupta and Waymire [1989] suggest a model for link drops given by

$$\frac{H(\alpha n)}{\mu(\alpha)} \overset{d}{=} H(n) \tag{1}$$

where $H(n)$ is the random link drop for a link of magnitude n, α is a scaling factor, $\mu(\alpha)$ is a normalization function, and $\overset{d}{=}$ denotes equality of probability distribution. Equation (1) is the definition of self-similarity, or scaling invariance of a random variable $H(n)$, dependent on and therefore indexed by scale parameter n. *Gupta and Waymire* [1989] show that quite generally $\mu(\ )$ is of the form

$$\mu(n) = n^{-\theta} \tag{2}$$

Note that here magnitude n is used as a surrogate for area, as discussed in Appendix B. *Gupta and Waymire* [1989] also suggest that link slopes are independent of link lengths. Our data, as will be shown below, indicates lack of correlation between link slopes and lengths, which justifies the independence assumption. The implication is that link drop is dependent on link length, which is also verified in the data. Therefore we prefer to start with a self-similar assumption, analogous to (1), for slopes:

$$S(\alpha n) \overset{d}{=} \mu(\alpha) S(n) \tag{3}$$

where $S(n)$ is link slope dependent on magnitude, with $\mu(\ )$ given by (2). The justification for this assumption is the notion of self-similarity, and the empirical observations of *Flint* [1974], and others, described in Appendix B, with the approximation of magnitude proportional to area (equation (A14)). Denote the random link length assumed independent of magnitude by L. In our data we have not been able to detect significant trends of L with magnitude. Then link drop is

$$H(n) = S(n)L \tag{4}$$

so

$$H(\alpha n) = S(\alpha n)L \overset{d}{=} \mu(\alpha)S(n)L = \mu(\alpha)H(n) \tag{5}$$

equivalent to (1).

Gupta and Waymire [1989] started with (1) and used the inverse argument to derive (3). Equation (3) can alternatively be stated as

$$Z = \frac{S(n)}{\mu(n)} \overset{d}{=} S(1) \tag{6}$$

i.e., the variable $S(n)/\mu(n)$ is an independent and identically distributed (iid) random variable (equal in distribution to $S(1)$) which we call Z. The self-similar model is simply

$$S(n) = \mu(n)Z$$
$$= n^{-\theta}Z \tag{7}$$

This clearly shows the nature of the self-similar model which is characterized by power law scaling, the scale parameter n is raised to the power $-\theta$.

Moments of the self-similar model scale proportionally to $(n^{-\theta})^k$, i.e.,

$$E[S(n)] = n^{-\theta}E(Z)$$
$$\text{Var}\,[S(n)] = n^{-2\theta}\,\text{Var}\,(Z) \tag{8}$$
$$M_k[S(n)] = n^{-k\theta}M_k(Z)$$

where $M_k(\ )$ denotes the kth moment.

The quantiles of the self-similar model all scale proportional to $n^{-\theta}$, i.e.,

$$QS_\gamma(n) = QZ_\gamma n^{-\theta} \tag{9}$$

where $QS_\gamma(n)$ is a quantile (nonrandom value) of the slope corresponding to a probability of exceedance γ, i.e.,

$$\text{Prob}\,[S(n) > QS_\gamma(n)] = \gamma \tag{10}$$

and similarly, QZ_γ is a quantile of Z:

$$\text{Prob}\,[Z > QZ_\gamma] = \gamma \tag{11}$$

Gupta and Waymire [1989] incorporated this scaling in terms of link drops, (5), into the random topology model. They computed the expected value of link concentration function (lcf), conditional on magnitude from this model, and showed that it compared well with empirically observed lcf's. This had been a problem in earlier efforts [*Gupta and Mesa*, 1988; *Gupta et al.*, 1986; *Mesa*, 1986].

The good fit to mean lcf's were the only results used by *Gupta and Waymire* [1989] to justify the model (equation (1)). We feel that comparisons to the mean lcf such as those of *Gupta and Waymire* [1989] are limited because only the mean trend of drop variation with magnitude is tested. Higher moment or quantile properties are not tested. Also using the mean lcf introduces unnecessary mathematical complications to test notions that can be tested directly. *Gupta and Waymire* [1989] emphasize the importance of the scaling invariance or self-similarity of the link drop probability distribution as providing a fundamental theoretical basis for some existing empirical relationships (see Appendix B, equation (A8)). Although they also suggest that multiscaling corrections may be necessary, based on the fact that the empirical scaling exponents reported by *Wolman* [1955] vary with quantile value, they did not explore nor attempt to explain the inconsistency of this observation with their self-similar model. In the next section we show that direct analysis of data does not support scaling invariance as a model for link slopes.

3. EMPIRICAL EVIDENCE OF SCALING IN ELEVATION

The data presented in this study is from Big Creek, a tributary of the St. Joe River near Calder, Idaho, and the St. Joe River itself. This data is part of an extensive empirical study over a wide range of areas (see *D. G. Tarboton*, manuscript in preparation, 1989). The results obtained are typical of all the areas we have studied. Big Creek is a 147×10^6 m^2 basin covered by a combination of four U.S. Geological Survey 7.5 min DEM's that give elevations on a 30-m grid. The St. Joe River is a 2834×10^6 m^2 basin with DEM data from a combination of three 1° Defense mapping agency series DEM's that give elevations on a 3 arc second grid (60 m × 90 m approximately). Figure 1 gives a map of the St. Joe River and Big Creek.

Figures 2 and 3 show link slopes plotted against magnitude, $2n - 1$ and area for these two data sets. In these figures n, $2n - 1$, and area are for practical purposes interchangeable scaling indices. We interpret area as being the fundamental scaling index, with n and $2n - 1$ good surrogate measures. In the remainder of the paper we use n as our scaling index, for simplicity and purposes of comparison with the work of *Gupta and Waymire* [1989]. We could have obtained the same results using $2n - 1$ or A.

In Figures 2 and 3 there is considerable scatter in the individual link slopes, however, by ranking the links according to scale index (A, n, or $2n - 1$), the links are grouped into bins containing at least 20 links that cover a narrow range of scale index. The group sample means are plotted as circles in Figures 2 and 3 and show power law scaling approximately proportional to $n^{-0.6}$. This by itself is in agreement with the self-similar model. The group sample variances are plotted in Figures 4 and 5 and show power law scaling proportional to $n^{-0.6}$. If the exponent θ is estimated from the mean (Figures 2 and 3) as 0.6, the self-similar model (equation (8)) would predict that slope variances should scale proportional to $n^{-1.2}$. This is not the case in Figures 4 and 5 and is the first indication of failure of the self-similar model.

Further evidence of the failure can be obtained by looking at the distribution of the normalized variable $Z = S(n)/\mu(n)$ (equation (6)). With $\theta = 0.6$ from Figures 2 and 3, we can compute Z for each link. This gives $N = 2n - 1$ realizations of Z, so for N large (which it is) we can get an idea of the probability distribution of Z by ranking and plotting according to the plotting position $P = i/(N + 1)$, where i is the rank from largest (1) to smallest (N) of the N realizations of Z. This is done in Figures 6 and 7 for the Big Creek and St. Joe Rivers. Also shown are the probability distributions estimated from subsets consisting of all links with a given Strahler order. The 95% confidence limit error bars in the plotting position probability computed from the incomplete beta distribution are shown. (The incomplete beta distribution is the theoretical distribution of nonparametric estimates of probability. For further information on this, see, for example *Zhang* [1982] and references therein.) The Z are supposed to be iid, i.e., independent of Strahler order, so sets of links from different Strahler orders should have the same probability distribution. This is not the case and is another indication of the failure of the self-similar model of slopes.

Tables 1 and 2 give statistics of link properties for the Big Creek and St. Joe Rivers. The tables show how the mean and variance of slope and drop both decreases with order. The normalization accounts for the trends in the mean, but

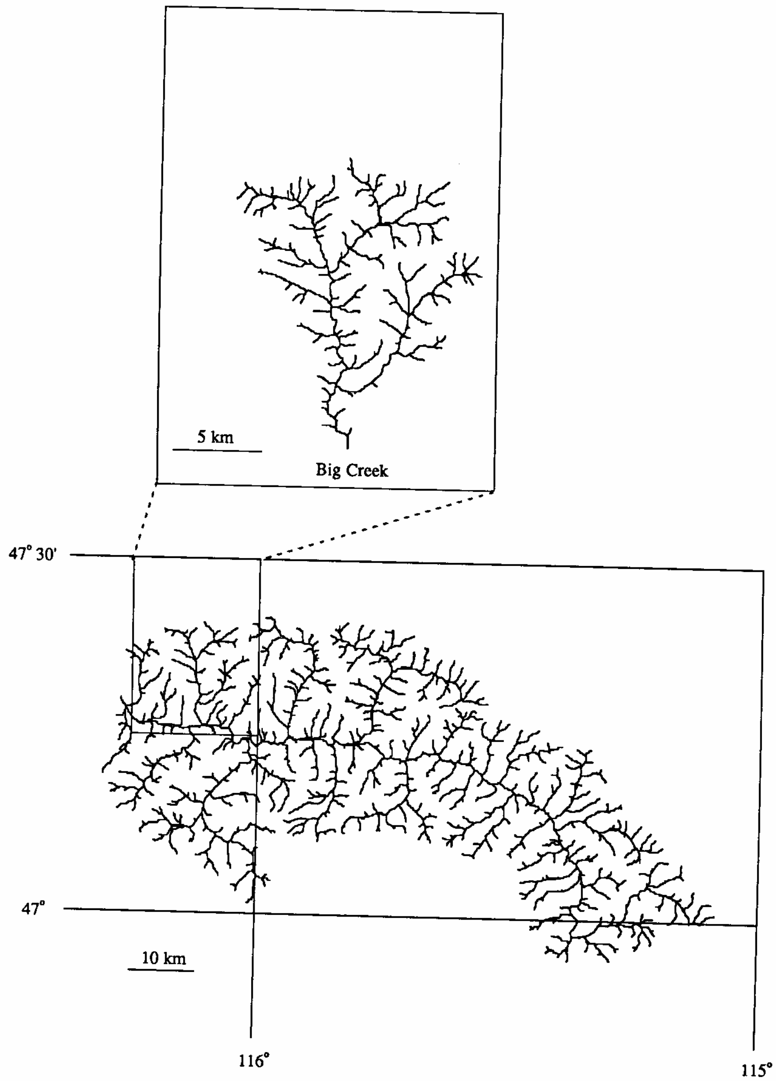

Fig. 1. St. Joe River and Big Creek location map.

makes the normalized variance and hence coefficient of variation increase with order, counter to what the self-similar model for full distribution would predict. Also, note that correlation coefficients between slope and length are negligible, whereas the correlation between drop and length is not. This gives some credence to the assumption of independence between slopes and lengths and is the basis for our regarding slope as the more fundamental variable than link drop in (3). Significance tests on the difference between the mean (t test) and variance (F test) of the normalized slopes given in Tables 1 and 2 were done. These indicate no significant differences between the mean normalized slope of links of different orders, indicating that the normalization works for the mean. However, for the variances the hypothesis that the different order links are from the same population is rejected at the 0.05 level for the great majority of cases. This is a clear failure of the self-similar model to characterize the link slope distribution.

4. Link Slope Scaling Model

Let us view the fall in elevation along a stream as composed of distinct discrete steps. This is consistent with the notion of pools and riffles due to *Yang* [1971*a*] and has possible justification in terms of energy expenditure arguments. The location of steps will be taken as random according to a general stationary point process along the length of channel. Thus the number of steps in a fixed length of channel will be a random variable. The size of individual steps will be taken as iid random variables, which implies that the accumulation of elevation changes is a marked point process. The scaling will be introduced through the intensity or rate of the point process λ, which will be taken proportional to $\mu(n) = n^{-\theta}$.

Let a link have a random length L. Then the number of steps in the link is a random variable J with distribution, conditional on L, denoted $P_{J|L}(j|l)$ for j a nonnegative

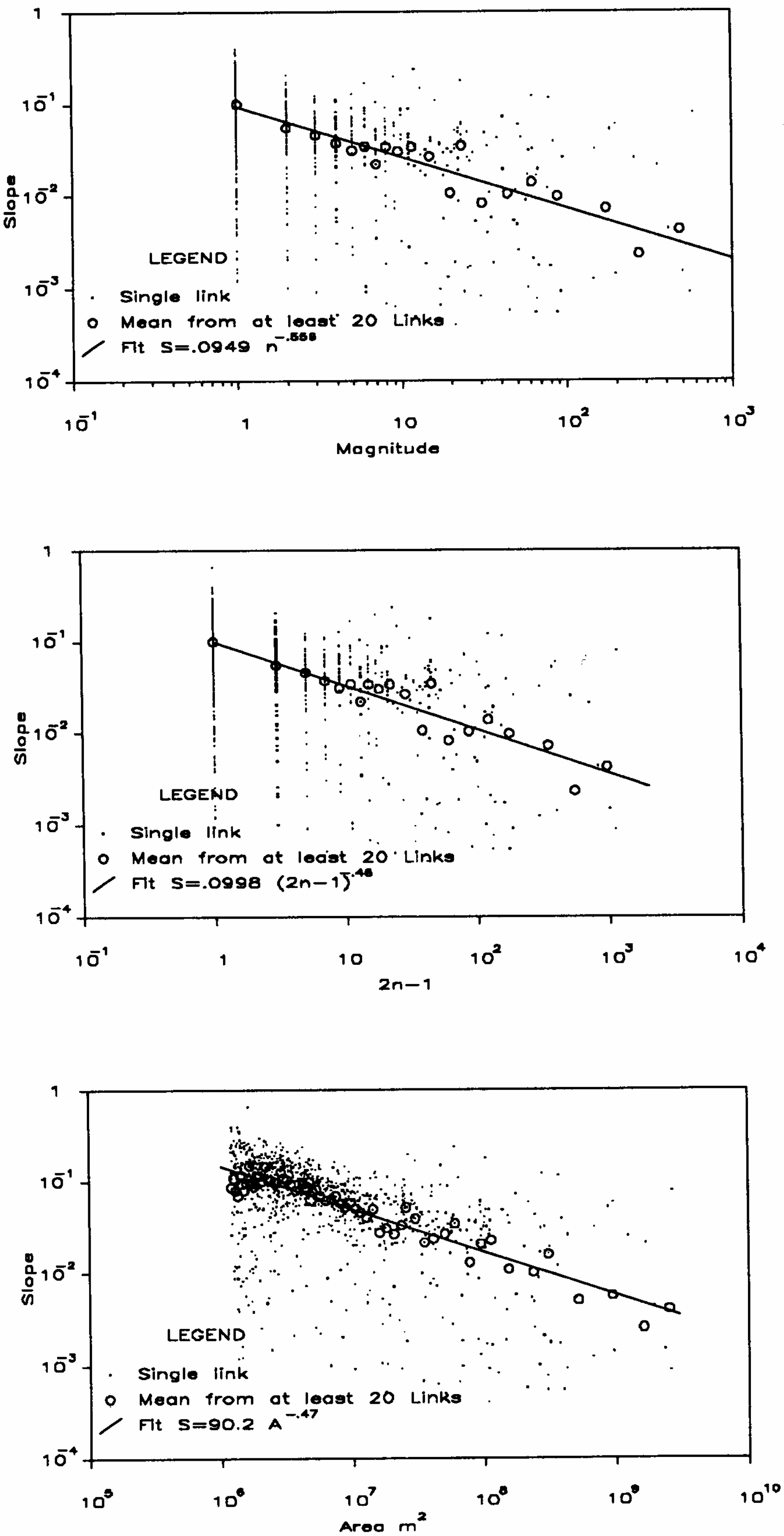

Fig. 2. Link slopes for St. Joe River, Idaho.

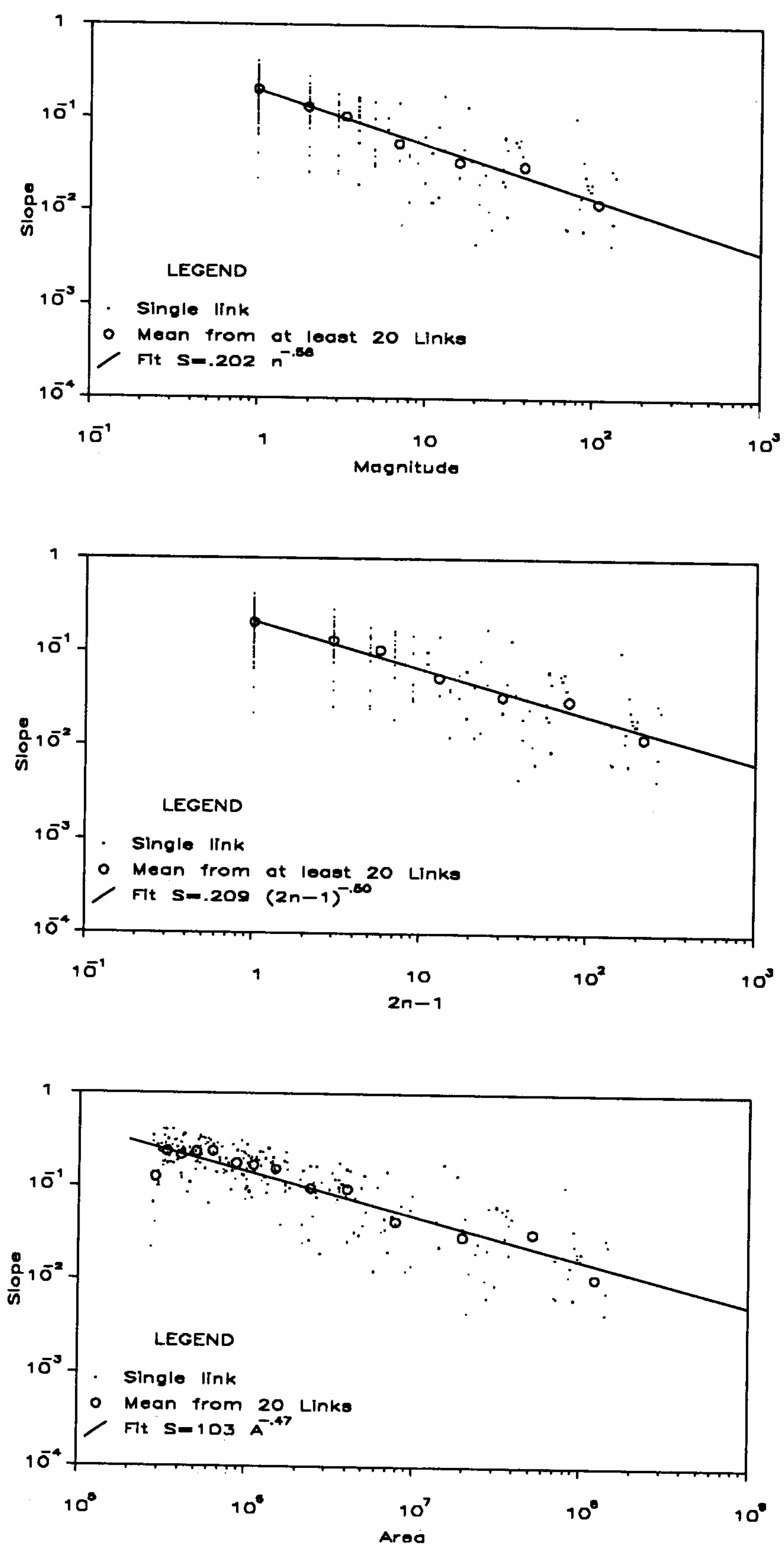

Fig. 3. Link slopes for Big Creek, Idaho.

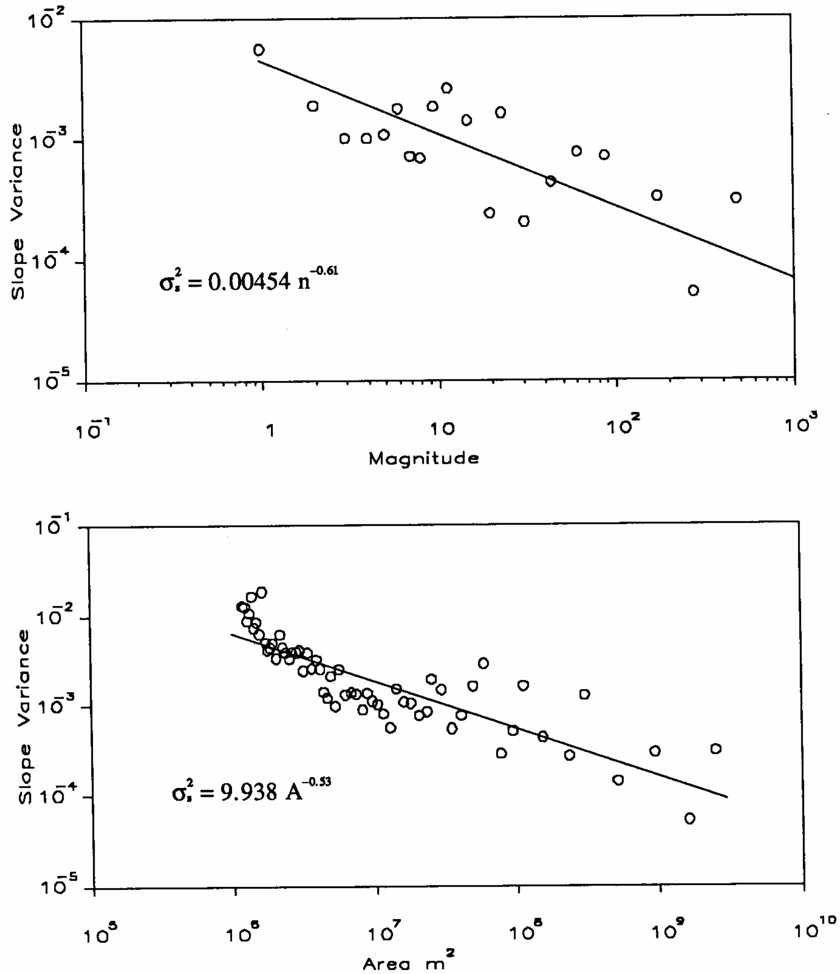

Fig. 4. Link slope variances, St. Joe River, Idaho.

integer (0, 1, 2, $\cdots$). From properties of generally orderly point processes [*Cox and Isham*, 1980], we get

$$E[J|L] = \lambda L \tag{12}$$

and we characterize the point process by its index of dispersion defined

$$I(L) = \frac{\mathrm{Var}\,[J|L]}{E[J|L]} \tag{13}$$

Note that for a Poisson process $I(L) = 1$, and for a process with uniform step spacing (i.e., no randomness or variance), $I(L) = 0$. The index of dispersion indicates overdispersion or underdispersion of the points with respect to the Poisson process, and in general may be scale dependent, i.e., dependent on L.

The steps are iid random variables denoted by Y_i, so the link drop is

$$H = \sum_{i=1}^{J} Y_i \tag{14}$$

Define the random link slope

$$S = H/L \tag{15}$$

The objective is to derive the moments of S and H, since these describe the random structure of the network in elevation. This is done in terms of the joint distribution of H and L. H is a function of J which is conditional on L, so H and L are not independent.

Let the Y_i have probability density function (pdf) $f_Y(y)$. Then for J given the pdf of H is the J fold convolution of $f_Y(y)$, *Feller* [1971], denoted $f_Y^{J*}(h)$. Summing this over the distribution of J we get, conditional on L

$$f_{H|L}(h|l) = \sum_{j=0}^{\infty} P_{J|L}(j|l) f_Y^{j*}(h) \tag{16}$$

From the definition of conditional probability, the joint pdf of H and L is

$$f_{H,L}(h, l) = f_{H|L}(h|l) f_L(l)$$

$$= f_L(l) \sum_{j=0}^{\infty} P_{J|L}(j|l) f_Y^{j*}(h) \tag{17}$$

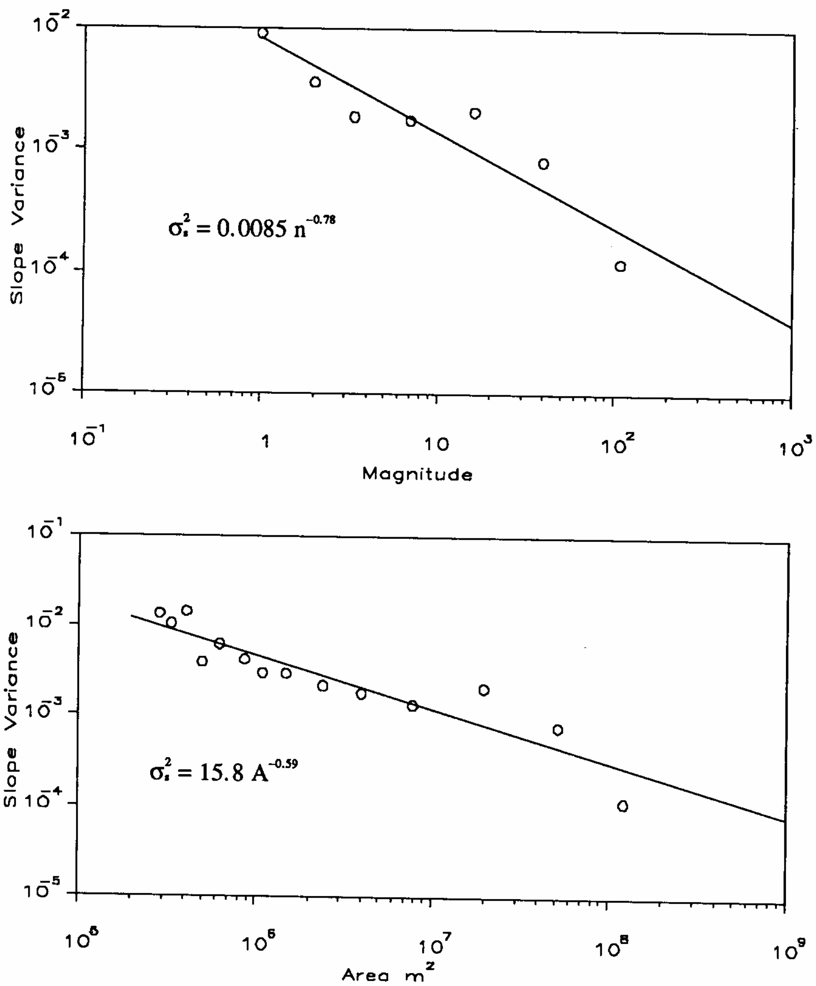

Fig. 5. Link slope variances, Big Creek, Idaho.

where $f_L(l)$ is the pdf of link lengths L. With this

$$E[S] = E\left[\frac{H}{L}\right] = \int_{h=0}^{\infty}\int_{l=0}^{\infty} \frac{h}{l} f_{H,L}(h, l)\, dh\, dl$$

$$= \int_{l=0}^{\infty} \frac{1}{l} f_L(l) \sum_{j=0}^{\infty} P_{J|L}(j|l)$$

$$\cdot \int_{h=0}^{\infty} h f_Y^{j*}(h)\, dh\, dl = \int_{l=0}^{\infty} \frac{1}{l} f_L(l)$$

$$\cdot \sum_{j=0}^{\infty} P_{J|L}(j|l) j\bar{Y}\, dl = \int_{l=0}^{\infty} \frac{1}{l} f_L(l) \lambda l \bar{Y}\, dl = \lambda\bar{Y} \quad (18)$$

Similarly,

$$E[S^2] = \int_{l=0}^{\infty}\int_{h=0}^{\infty} \frac{h^2}{l^2} f_{H,L}(h, l)\, dh\, dl$$

$$= \lambda\left\{\sigma_Y^2 E\left[\frac{1}{Y}\right] + \bar{Y}^2 E\left[\frac{I(L)}{L}\right]\right\} + \lambda^2\bar{Y}^2 \quad (19)$$

where σ_Y is the standard deviation of the step height Y and the expectation is now over the distribution of link length. Then

$$\mathrm{Var}\,[S] = E[S^2] - \{E[S]\}^2 = \lambda\{\sigma_Y^2 E[1/L] + \bar{Y}^2 E[I(L)/L]\} \quad (20)$$

Similarly, we obtain

$$E[H] = \lambda\bar{L}\bar{Y} \quad (21)$$

$$\mathrm{Var}\,[H] = \sigma_L^2\lambda^2\bar{Y}^2 + \lambda\{\sigma_Y^2\bar{L}^2 E[LI(L)]\} \quad (22)$$

$$\mathrm{Cov}\,(S, L) = \mathrm{Corr}\,(S, L) = 0 \quad (23)$$

$$\mathrm{Corr}\,(H, L) = \left[1 + \frac{E[LI(L)]/\bar{L} + \sigma_Y^2/\bar{Y}^2}{\lambda\bar{L}(\sigma_L^2/\bar{L}^2)}\right]^{-1/2} \quad (24)$$

where Cov denotes covariance and Corr denotes correlation. Note that the correlation between S and L is 0 as observed in Tables 1 and 2, whereas the correlation between H and L is not. Implicit in these results is the assumption that expectations $E[I(L)/L]$, $E[LI(L)]$, and $E[1/L]$ exist. This places minor restrictions on the distribution that can be used for L and the point process that can be used. Now the scaling can be introduced by letting

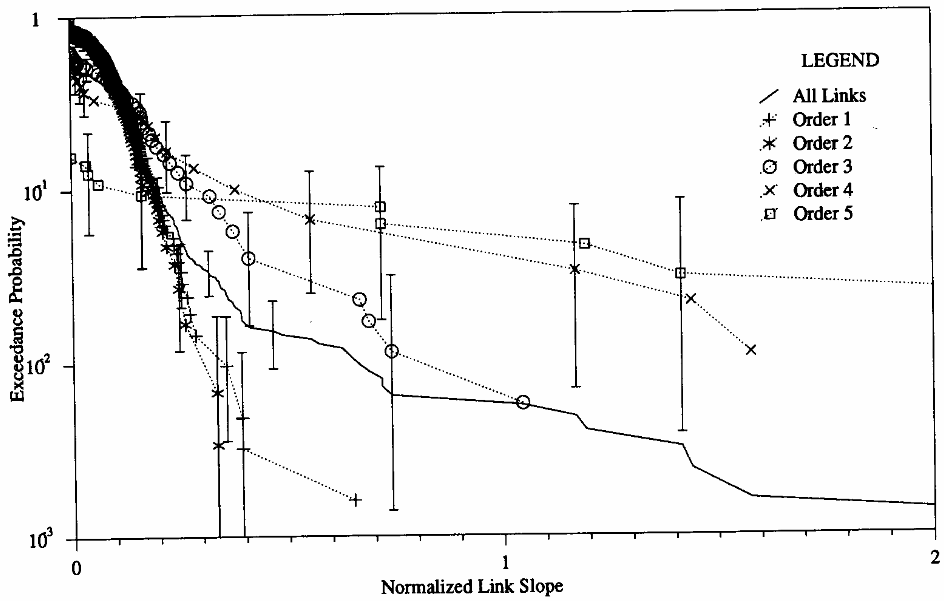

Fig. 6. St. Joe River link slopes normalized with $n^{-0.6}$.

$$\lambda = k\mu(n) = kn^{-\theta} \tag{25}$$

From (18) we then get

$$E[S] = k\bar{Y}n^{-\theta} \tag{26}$$

and from (20)

$$\text{Var }[S] = k\{\sigma_Y^2 E[1/L] + \bar{Y}^2 E[I(L)/L]\}n^{-\theta} \tag{27}$$

This is of the form observed in Figures 2–5 and is different from the scaling predicted by the self-similar slope model of *Gupta and Waymire* [1989]. From (26) and (27), we get the coefficient of variation of slope

$$C_s = \frac{(\text{Var }[S])^{1/2}}{E[S]} = \left\{\frac{(\sigma_Y^2/\bar{Y}^2)E[1/L] + E[I(L)/L]}{k}\right\}^{1/2} n^{1/2} \tag{28}$$

which is an increasing function of n or order, as observed in Tables 1 and 2.

The elements of this model essential to explain the multiscaling of link slopes are the Y_i being iid and the mean density of steps (or rate of the point process) being proportional to $n^{-\theta}$. For a fixed length of channel x, and letting λ be proportional to $n^{-\theta}$ in (18) and (20), we get

$$E[S] \sim n^{-\theta}\bar{Y} \tag{29}$$

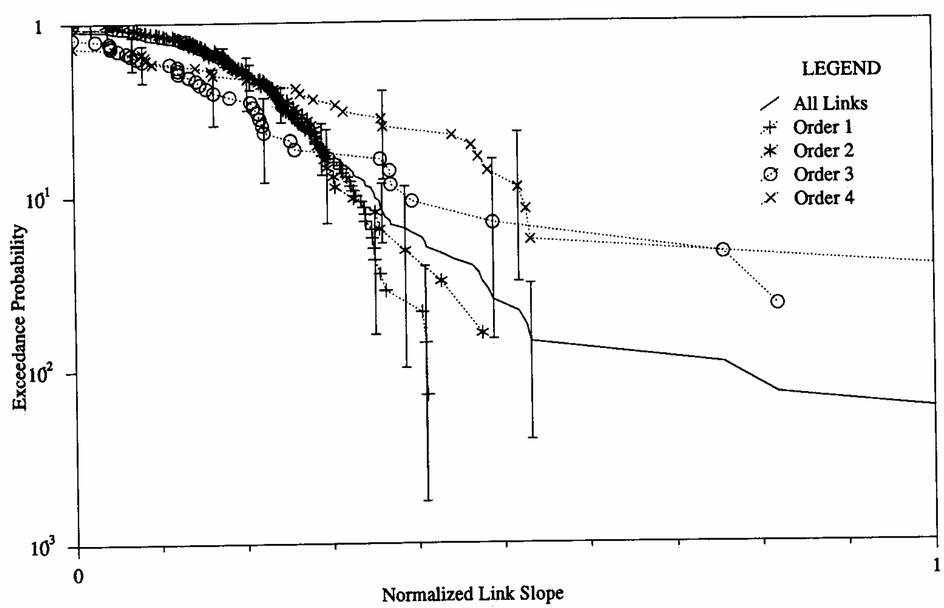

Fig. 7. Big Creek link slopes normalized with $n^{-0.6}$.

TABLE 1. Big Creek Link Statistics (Magnitude 139, Order 4 Basin)

	All Orders Combined	Strahler Order 1	2	3	4
Mean					
H(m)	82	121	79	21	9.4
S	0.136	0.203	0.114	0.039	0.020
L(m)	614	591	735	612	497
Variance					
$H(10^3\ m^2)$	7.36	9.0	3.6	0.54	0.12
$S(10^{-3})$	11.1	9.2	2.9	1.9	0.55
$L(10^3\ m^2)$	240	221	228	384	141
Coefficient of variation					
H	1.04	0.79	0.77	1.10	1.16
S	0.78	0.47	0.48	1.12	1.20
L	0.80	0.79	0.65	1.01	0.76
Normalized mean					
$H/\mu(n)$(m)	123	121	145	102	117
$S/\mu(n)$	0.204	0.203	0.208	0.173	0.238
Normalized variance					
$H/\mu(n)(10^3\ m^2)$	11.9	9.0	12.6	16.2	16.7
$S/\mu(n)(10^{-3})$	22	9.2	9.0	34.7	78.1
Coefficient of variation for normalized					
$H/\mu(n)$	0.89	0.79	0.77	1.24	1.10
$S/\mu(n)$	0.72	0.47	0.46	1.07	1.17
Correlation coefficient between					
H and L	0.61	0.84	0.66	0.77	0.55
S and L	−0.02	0.009	−0.20	−0.09	−0.02
normalized H and L	0.77	0.84	0.67	0.85	0.59
normalized S and L	−0.04	0.009	−0.17	−0.03	−0.01

Number of links for each order: 1, 139; 2, 61; 3, 42; 4, 35; total 277. H, link drop; L, link length; and S, link slope defined as H/L. The function $\mu(n) = n^{-0.6}$ is divided into H or S to get the normalized statistics.

$$\mathrm{Var}\,[S] \sim n^{-\theta}/x\{\sigma_Y^2 + \bar{Y}^2 I(x)\} \tag{30}$$

In summary, in constructing models of link drops or slopes, there are two possibilities. The normalization $n^{-\theta}$ can be applied to the step height Y or step density λ. We see that the choice results in fundamentally different scaling behavior. Applying the normalization to step height Y leads to moments scaling proportional to $(n^{-\theta})^k$ and the self-similar model. Applying the normalization to the density λ leads to moments scaling proportional to $n^{-\theta}$ as observed. An important finding is therefore that the scaling of slopes is consistent with self-similarity in the density of steps or rate of elevation changes and not with simple self-similarity of the slopes or drops of individual links. This pinpoints the cause and clarifies the nature of the multiscaling of channel slopes. A physical explanation of why the density of steps or elevation increments scale the way suggested here is an open question.

The discreteness of the steps in this model can be removed by considering a limit process with $k \to \infty$ and $\bar{Y} \to 0$ that maintains the desired properties. In particular, we must have $E[S]$ given by (26) remain bounded, i.e., not diverge to ∞ or 0. This suggests a limit in which

$$\lim_{\substack{k\to\infty\\ \bar{Y}\to 0}} (k\bar{Y}) = \text{const} \tag{31}$$

i.e., $k \sim 1/\bar{Y}$. Also, Var $[S]$ as given by (27) must remain bounded. The variance can be written as

$$\mathrm{Var}\,[S] = k\bar{Y}^2\{E[1/L]C_Y^2 + E[I(L)/L]\}n^{-\theta} \tag{32}$$

where $C_Y = \sigma_Y/\bar{Y}$. From (31),

$$\lim_{\substack{k\to\infty\\ \bar{Y}\to 0}} (k\bar{Y}^2) = 0$$

so we must have

$$\lim_{\substack{k\to\infty\\ \bar{Y}\to 0}} (k\bar{Y}^2 C_Y^2) = \text{const} \tag{33}$$

With (31) this implies

$$\lim_{\substack{k\to\infty\\ \bar{Y}\to 0}} (\bar{Y}C_Y^2) = \lim_{\substack{k\to\infty\\ \bar{Y}\to 0}} (\sigma_Y^2/Y) = \text{const} \tag{34}$$

The question is whether distributions exist that have $\bar{Y} \to 0$, $\sigma_Y^2 \to 0$, but $C_Y \to \infty$ as is needed by (34). One possibility is the gamma distribution, which also has the desirable property that $Y > 0$ as is sensible for link drops. The gamma pdf is [*Feller*, 1971]

$$f_Y(y) = \frac{\beta(\beta y)^{\nu-1}e^{-\beta y}}{\Gamma(\nu)} \tag{35}$$

where β is the scale parameter and ν the shape parameter. This has moments

$$\bar{Y} = \nu/\beta \qquad \sigma_Y^2 = \nu/\beta^2 \tag{36}$$

which give

$$C_Y = 1/\sqrt{\nu} \tag{37}$$

Therefore

$$C_Y^2\bar{Y} = 1/\beta \tag{38}$$

$$k\bar{Y} = k\nu/\beta \tag{39}$$

so a gamma distribution with β held constant and $\nu \sim 1/k$ will satisfy our conditions (34) and (31) in the limit as $k \to \infty$.

This limit process is analogous, but not equivalent to the limit of a random walk resulting in Brownian motion. In Brownian motion the central limit theorem gives the Gaussian distribution as the limit distribution. The central limit theorem [*Feller*, 1971, p. 259] applies to the limit sum of mutually independent random variables with common distribution. Here the limit involves changing the shape of the distribution as the limit is approached. Clearly, the central limit theorem does not apply. In fact when Y is gamma distributed (equation (35)) and we consider a fixed length of channel L with uniform step distribution ($I(L) = 0$)), the number of steps is $kn^{-\theta}L$ so H and S are gamma distributed. For S,

TABLE 2. St. Joe River Link Statistics (Magnitude 621, Order 5 Basin)

	All Orders Combined	Strahler Order 1	2	3	4	5
Mean						
H(m)	90.6	135	72.8	30.5	14.9	4.8
S	0.066	0.100	0.047	0.026	0.013	0.0038
L(m)	1356	1383	1465	1233	1180	1176
Variance						
$H(10^3\ m^2)$	11.2	14.7	5.1	1.27	0.53	0.26
$S(10^{-3})$	4.7	5.6	1.4	1.3	0.78	0.20
$L(10^3\ m^2)$	1209	1527	1054	785	616	675
Coefficient of variation						
H	1.17	0.90	0.98	1.17	1.55	3.33
S	1.04	0.75	0.82	1.41	2.08	3.66
L	0.81	0.89	0.70	0.72	0.66	0.70
Normalized mean						
$H/\mu(n)$(m)	136	135	135	131	145	149
$S/\mu(n)$	0.102	0.100	0.088	0.109	0.13	0.126
Normalized variance						
$H/\mu(n)(10^3\ m^2)$	32.6	14.8	16.9	24.8	65.1	262
$S/\mu(n)(10^{-3})$	26.3	5.6	5.1	24.8	78.5	264
Coefficient of variation for normalized						
$H/\mu(n)$	1.33	0.90	0.96	1.20	1.75	3.43
$S/\mu(n)$	1.59	0.74	0.81	1.44	2.16	4.08
Correlation coefficient between						
H and L	0.67	0.79	0.73	0.61	0.32	0.16
S and L	0.01	−0.03	0.09	−0.03	−0.04	0.02
normalized H and L	0.51	0.78	0.74	0.57	0.28	0.14
normalized S and L	−0.01	−0.03	0.09	−0.03	−0.04	0.004

Number of links for each order: 1, 621; 2, 295; 3, 173; 4, 89; 5, 16; total 1241. H, link drop; L, link length; and S, link slope defined as H/L. The function $\mu(n) = n^{-0.6}$ is divided into H or S to get the normalized statistics.

$$f_{S|L}(s|l) = \frac{\beta l(\beta sl)^{kC\nu-1}e^{-\beta sl}}{\Gamma(kC\nu)} \tag{40}$$

where $C = n^{-\theta}l$.

In the limit $\nu \to 0$, $k \to \infty$, but $\nu \sim 1/k$, i.e., take $\nu k = C_2$ a constant. Therefore the pdf of slope for a fixed length of channel is by construction, for all k including the limit $k \to \infty$, the gamma distribution

$$f_{S|L}(s|l) = \frac{\beta l(\beta sl)^{CC_2-1}e^{-\beta sl}}{\Gamma(CC_2)} \tag{41}$$

In Appendix C we show that this is also the limit distribution for all point processes that are asymptotically normal.

Empirical support for the link slope scaling model suggested in this section is obtained by looking at the probability distribution of link slopes. The unconditional or marginal distribution for link slope is

$$f_S(s) = \int_{l=0}^{\infty} f_L(l)f_{S|L}(s|l)\,dl \tag{42}$$

A common although perhaps not the best distribution for link lengths is the gamma distribution, written here (analogous to equation (35))

$$f_L(l) = \frac{1}{\mu}\frac{(l/\mu)^{\alpha-1}e^{-l/\mu}}{\Gamma(\alpha)} \tag{43}$$

See L. D. van der Tak and R. L. Bras, (Stream length distributions and hillslopes effects in the geomorphologic IUH, submitted to *Water Resources Research*, 1989), *Abrahams* [1984], or *Abrahams and Miller* [1982] for discussion of the merits of various link length distributions.

We were not able to evaluate the integral in (42). This at first glance appears simply the product of gamma functions; however, the realization that $C = n^{-\theta}l$ in (41) complicates the integration over l. Hence we estimated $f_S(s)$ by simulation. First, we sample from the distribution of L (equation (43)), and then with L known from the distribution of $S|L$ (equation (41)).

Figures 8 and 9 gives estimates for the exceedance probability distribution of link slopes. The parameters used were $\alpha = 1.96$, $\mu = 313.7$, $\beta = 0.04225$, $C_2 = \nu k = 0.0086$, $\theta = 0.6$, and λ given by (25). These parameters were chosen to match the following moments of the Big Creek data set: $E[L]$, $E[1/L]$, $E[S]$, Var $[S]$. One thousand variates were simulated and the exceedance probability obtained from the plotting position $P = i/(N + 1)$. The magnitudes of 1, 4, 16, and 64 were chosen because according to the random

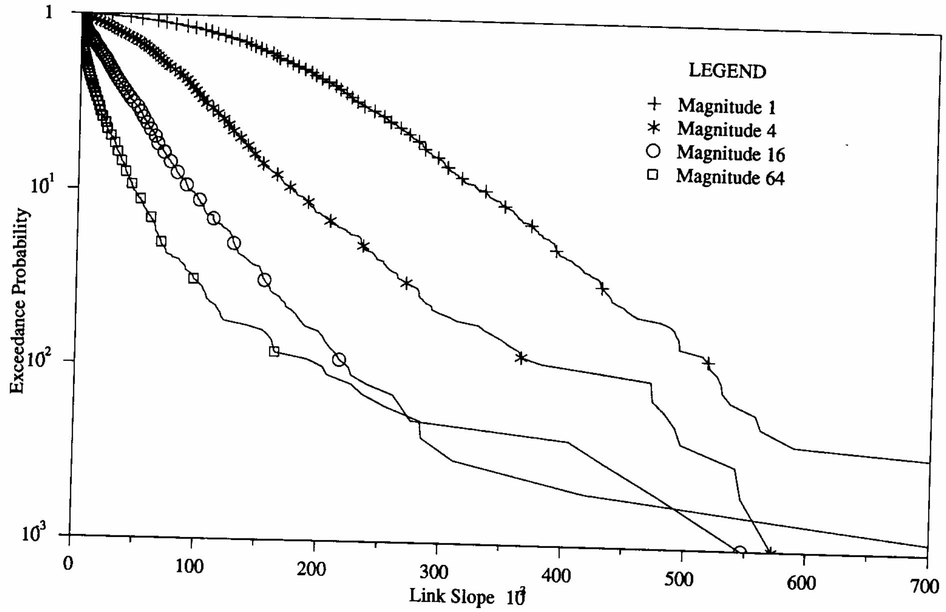

Fig. 8. Model slope distributions by simulation.

topology model in infinite networks they are equivalent to Strahler orders 1, 2, 3, and 4. In Figure 8 note the decrease in mean slope with order as well as change in shape emphasizing the non-self-similarity of the slope distributions. Figure 9 gives the same data but here the slopes have been normalized by $\mu(n) = n^{-\theta}$. The mean is preserved but the distributions differ in shape. The similarity between simulated (Figure 9) and empirical (Figure 6) distributions is evident. In Figure 10 the link slope data for Big Creek are plotted without normalization (contrary to Figure 4). The comparison of the simulated link slope distributions (Figure 8) and the empirical distributions for Big Creek (Figure 10) is very good.

5. CONCLUSIONS

The important conclusions from this work are as follows.

1. Link slopes are not self-similar with respect to area as a scale index. Rather they are multiscaling, manifested by the fact that the coefficient of variation of slope increases with scale (i.e., area or magnitude).

2. The nature of this multiscaling has been identified. It is such that the density of elevation increments within a channel scales proportional to $A^{-\theta}$ (or $n^{-\theta}$) while the increments themselves show no apparent trend.

3. The data has confirmed that link slope and link length are uncorrelated, although not necessarily independent. This

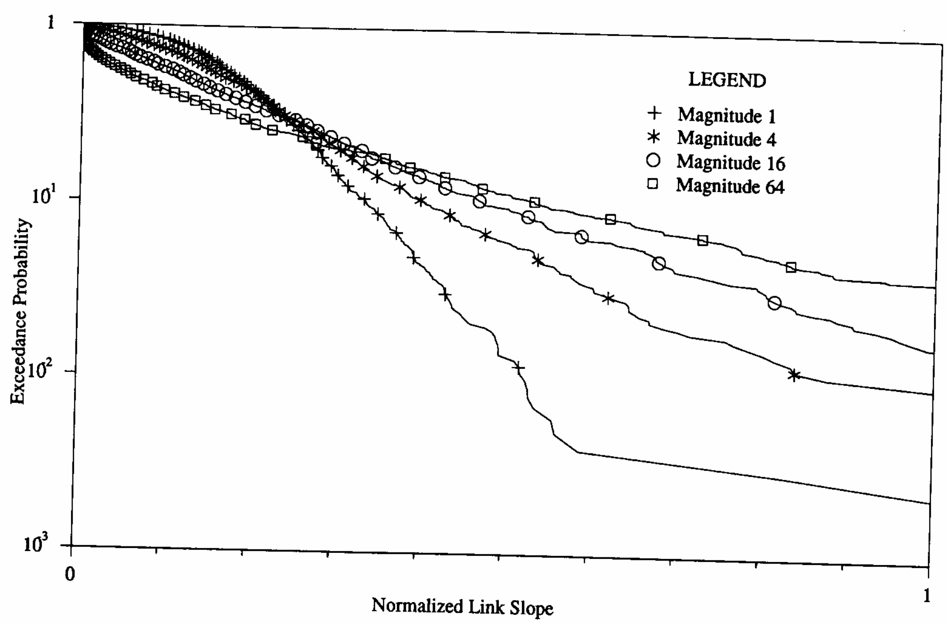

Fig. 9. Normalized slope distributions from model by simulation.

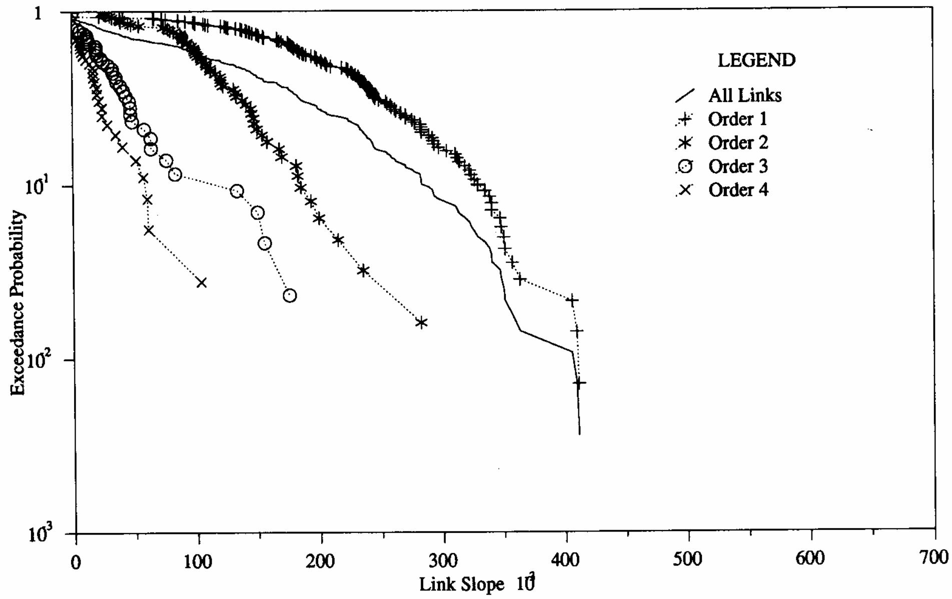

Fig. 10. Big Creek link slopes (not normalized).

implies that link drop is correlated with link length and that therefore slope is a more fundamental variable than drop.

Although the data presented is for only two river basins, the conclusions are believed to be generally valid for the majority of rivers in a state of balance or equilibrium. We are obtaining (D. G. Tarboton, manuscript in preparation, 1989) results for many more data sets that are qualitatively the same although the scaling exponent θ seems to be slightly variable. Further work is required to discover the physical mechanisms that result in these observations.

Appendix A: Channel Networks From Digital Elevation Models

The procedures for obtaining channel networks are based on work of *O'Callaghan and Mark* [1984] and *Band* [1986] and give results essentially equivalent to the procedures of *Carrera* [1988]. The DEM's consist of elevations in a rectangular grid with, usually, 30 m spacing. Each rectangular grid block is termed a pixel and the first step in determining drainage network is to assign a drainage direction from each pixel to one of its eight neighbors (including diagonals). This is done in the direction of steepest descent. Where the slope in two or more directions is the same, as is typically the case in a flat area, the directions are assigned arbitrarily, but such that no loops are formed. At this stage it is also necessary to resolve data anomalies, generally manifested as pits lower than all surrounding pixels. These are regarded as data errors since in a fluvial landscape, pits large enough to be resolved on a 30-m grid are rarely seen in practice. The pits are resolved by filling them, i.e., increasing the elevation until they drain, meaning that consistent drainage directions with nonnegative slopes can be determined. These adjustments are relatively minor and typically are only required for a small subset (<2%) of the pixels. The adjustments are on average smaller than the root-mean-square error of the data determined by the U.S. Geological Survey.

The next step is to count the number of pixels that drain through each pixel. This provides a measure of convergence of flow, with the largest accumulation of area being in the valleys along the streams. This suggests defining streams as those pixels with total drainage area greater than a support area threshold. This is conceptually similar to the constant of channel maintenance notion [*Schumm*, 1956]. *Carrera* [1988] uses a support area threshold as well as minimum source stream length to define streams.

The setting of directions and accumulation of areas is done for the whole of a rectangular DEM data set. Isolation of the drainage basin consists of identifying those pixels that eventually drain through an outlet pixel. The channel network is defined as the geometric tree of lines along flow directions joining the centers of all pixels with accumulation area above the support area threshold. The terminology used is fairly standard and basically that of *Shreve* [1966, 1967]. Sources are the points upstream on this network and junctions are points where two (or more) channels join. Exterior links are defined to be segments of the channel network between a source and the first junction downstream, and interior links are segments between two successive junctions or between the outlet and first junction upstream.

The magnitude n of a network is the number of sources or exterior links in the network. The magnitude of a link is the magnitude of the network draining into that link. The networks can be described by the *Strahler* [1952] ordering system. This is as follows: all exterior links have order 1. Where two or more links of order $m_1, m_2, m_3, \cdots$ join with $m_1 \geq m_2 \geq m_3, \cdots$, the order of the next downstream link is max $(m_1, m_2 + 1)$. This is a slight modification of the usual system to allow for the fact that on a discrete grid junctions of more than two links are possible. A Strahler stream is a sequence of links of the same order.

The most appropriate choice of support area threshold to use to define the network is the subject of ongoing research. The details are not important to the present topic, except to note that the support area thresholds used are what we believe are the most appropriate and give the correct drainage density for the data set used. Once the network is determined, it is straightforward to compute the properties of each link, i.e., magnitude, drop, order, length. Slope is defined as drop/length, i.e., is an average over the link. Area is measured at the downstream end of each link and is the area of the whole subbasin draining into each link.

APPENDIX B: LITERATURE REVIEW

The first empirical characterization of elevation and slope properties was *Horton*'s [1945] slope law:

$$R_s = S_w/(S_{w+1}) \quad \text{(A1)}$$

Here R_s is the slope ratio and S_w, S_{w+1} the mean slope of streams of order w, $w + 1$, respectively. Practically, R_s is obtained from a least squares fit to plots of log S_w versus w and typically has values between 1.5 and 3. Horton's other laws similar to (A1), for stream lengths, areas, and numbers are

$$R_L = L_w/(L_{w-1}) \quad \text{(A2)}$$

$$R_a = A_w/(A_{w-1}) \quad \text{(A3)}$$

$$R_b = (N_{w-1})/N_w \quad \text{(A4)}$$

where L_w and A_w are the mean length and area draining streams of order w, and N_w is the number of streams of order w, for any integer $w > 0$.

Note that the slope law (A1) implies exponential scaling of mean slope with order

$$S_w = (R_s S_1) R_s^{-w} = (R_s S_1) e^{-w \ln R_s} \quad \text{(A5)}$$

Power law relationships have been widely used in the hydrologic/geomorphologic literature to describe the scaling of hydraulic-geometric variables [*Wolman*, 1955; *Leopold et al.*, 1964; *Leopold and Miller*, 1956; *Flint*, 1974]. One such relation is

$$S = tQ^z \quad \text{(A6)}$$

where S is slope, Q is discharge, and t and z are coefficients.

A common relationship between discharge and area [*Flint*, 1974; *Leopold and Miller*, 1956; *Leopold et al.*, 1964] is

$$Q = aA^x \quad \text{(A7)}$$

where A is basin area and a and x are coefficients.

Combining these, *Flint* [1974] gives

$$S = ta^z A^{xz} \quad \text{(A8)}$$

Grouping the coefficients, leads to the fundamental scaling described by *Gupta and Waymire* [1989], that is,

$$E[S(A)] = \text{const } A^{-\theta} \quad \text{(A9)}$$

The expectation $E[\;\;]$ is used because *Gupta and Waymire* [1989] regard slope as a random variable characterized by a scaling function of the form

$$\mu(A) = A^{-\theta} \quad \text{(A10)}$$

with area playing the role of a scaling index.

In *Flint*'s [1974] work the exponent $\theta = -xz$ takes on values with an average of 0.60 and a range of 0.37–0.83. A connection between θ and Horton's laws is also provided by *Flint* [1974]:

$$\theta = \frac{\ln R_s}{\ln R_a} \quad \text{(A11)}$$

This can be obtained by using the area law (A3) in (A9). This connects the exponential scaling with basin order described by Horton's slope law and the power law scaling with area present in (A9).

A connection with magnitude is seen using *Shreve*'s [1967] relationship between area and magnitude

$$A = A_1'(2n - 1) \quad \text{(A12)}$$

Here A_1' is the mean area draining directly into a link and $2n - 1$ is the total number of links. In (A9) this gives

$$E[S(A)] = E[S(n)] = \text{const } (2n - 1)^{-\theta} \quad \text{(A13)}$$

Flint [1974] actually estimated θ by a regression of log S versus log $(2n - 1)$. For n large (A13) can be written

$$E[S(n)] = \text{const } n^{-\theta} \quad \text{(A14)}$$

where the -1 is neglected and $2^{-\theta}$ is incorporated in the constant. This power law scaling of slope with magnitude is the form used by *Gupta and Waymire* [1989].

By using *Horton*'s [1945] bifurcation law (A4), we can estimate the number of first-order streams (i.e., magnitude) in a subbasin of order w as

$$n = R_b^{w-1} \quad \text{(A15)}$$

Using this in (A14) gives analogously to (A11)

$$\theta = \frac{\ln R_s}{\ln R_b} \quad \text{(A16)}$$

This connects the exponential scaling with basin order (equation (A5)) and the power law scaling represented by (A14). Equation (A15) basically shows that log n and order w are equivalent measures of network scale, as suggested by *Shreve* [1967].

There is a discrepancy between (A16) and (A11) in the fact that R_b cannot equal R_a in a given finite channel network. This arises from the approximations $n \sim A$ and the fact that the three equations (A3), (A4), and (A12) with $A_1' = A_1$ are mathematically inconsistent. Taking any two of them to hold exactly contradicts the third. This is not of practical importance, since these results are all empirical observations that do not hold exactly in nature.

The slope scaling and concave longitudinal stream profiles are often attributed to notions of dynamical equilibrium [*Langbein and Leopold*, 1964]. The slope scaling is often quantified and "explained" in terms of minimum work, minimum entropy principles [*Langbein*, 1964; *Leopold and Langbein*, 1962; *Scheidegger*, 1964; *Yang*, 1971*b*]. While the mathematical validity of these results is in doubt [*Kennedy et al.*, 1965], we feel that we agree qualitatively with the idea that the widespread observation of slope scaling is no accident probably explainable by a (to date unknown) physical principle. The apparent self-similarity of landscapes [*Mandelbrot*, 1983; *Mark and Aronson*, 1984] may be related. Note that the slope scaling as described by Horton's slope law, or the power law (equation (A8)) cannot extend indefinitely as it would imply infinite slopes at small scales. A break in scaling at scales smaller than about 0.6 km was observed by *Mark and Aronson* [1984]. We (D. G. Tarboton, manuscript in preparation, 1989), are using a break in slope scaling to identify the fundamental scale, or drainage density of river networks. The fundamental scale is seen as a scale where domination of sediment transport changes from fluvial processes to nonfluvial diffusion processes. *Andrle and Abrahams* [1989] working at scales less than 10 m on talus slopes also identify fundamental scales, associated with the size of boulders. It appears therefore that there are at least two fundamental scales in the landscape: particle/boulder size (0.001–10 m) and hillslope scale or drainage density (10–10^4 m). The precise value of these scales will vary for different landscapes dependent on geological and climatic conditions. Between these scales there is a range of scaling characterized by diffusive degradation [see *Culling*, 1986] and above the hillslope scale the landscape scaling is characterized by the stream slope scaling discussed here.

APPENDIX C: LIMIT SLOPE DISTRIBUTION FOR GENERALIZED POINT PROCESS

Here we use a result from statistical mechanics, referred to as the saddle point approximation or method of steepest descent, to show that when a gamma distribution is assumed for increment heights, the limit slope distribution conditional on length is also the gamma distribution given in (41), for all point processes that are asymptotically normal. This is a wide class of point processes including the Poisson process, renewal processes, and Neyman Scott processes [*Cox and Isham*, 1980]; we do not believe that the asymptotic normal assumption is particularly restrictive.

Equation (35) in (16) gives for fixed L with $\nu = C_2/k$ and the change of variables $H = SL$

$$f_{S|L}(s|l) = \sum_{j=0}^{\infty} P_{J|L}(j|l)\, \frac{\beta l(\beta s l)^{C_2 j/k - 1}}{\Gamma(jC_2/k)}\, e^{-\beta s l} \quad \text{(A17)}$$

Now substituting the normal density function for $P_{J|L}(j|l)$ we get

$$f_{S|L}(s|l) = \int_{-\infty}^{\infty} \frac{1}{(2\pi kC)^{1/2}}\, e^{-(j-kC)^2/2kC} \cdot \left[\frac{\beta l(\beta s l)^{C_2 j - 1}}{\Gamma(jC_2/k)}\right] e^{-\beta s l}\, dj = \int_{-\infty}^{\infty} e^{k\varphi(j)}\, dj \quad \text{(A18)}$$

where

$$\varphi(j) = -\frac{(j-kC)^2}{2k^2C} + \frac{\ln \beta l}{k} + \left(\frac{C_2 j - k}{k^2}\right)\ln \beta s l - \frac{\beta s l}{k} - \frac{1}{k}\ln (2\pi kc)^{1/2} - \frac{1}{k}\ln \Gamma\left(\frac{jC_2}{k}\right) \quad \text{(A19)}$$

Theorem

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} e^{k\varphi(x)}\, dx = e^{k\varphi(x_m)} \quad \text{(A20)}$$

where this limit exists. Here x_m is the value of x that corresponds to the global maximum of $\varphi(x)$. Clearly, for the limit to exist, $\varphi(x)$ must have highest-order k terms proportional to k^{-1} except for isolated x. An integral analogous to (A20) occurs in statistical mechanics.

Proof

This proof repeated here for convenience can also be deduced from *Negele and Orland* [1987, p. 121] or *Huang* [1963, p. 210].

Expand $\varphi(x)$ about x_m in a Taylor series

$$\varphi(x) = \varphi(x_m) + (x - x_m)\varphi'(x_m) + \frac{(x-x_m)^2}{2!}\varphi''(x_m) + \sum_{n=3}^{\infty} \frac{(x-x_m)^n}{n!}\varphi^{(n)}(x_m) \quad \text{(A21)}$$

Since $\varphi(x_m)$ is a maximum $\varphi'(x_m) = 0$ and $\varphi''(x_m) < 0$, we can write

$$I = \int_{-\infty}^{\infty} e^{k\varphi(x)}\, dx = \int_{-\infty}^{\infty} \exp\left(k\varphi(x_m) - \frac{k}{2}(x-x_m)^2|\varphi''(x_m)| + \sum_{n=3}^{\infty} k\,\frac{(x-x_m)^n}{n!}\varphi^{(n)}(x_m)\right) dx \quad \text{(A22)}$$

With the change of variables $\tau = (k|\varphi''(x_m)|)^{1/2}(x - x_m)$ this becomes

$$I = \frac{e^{k\varphi(x_m}}{(k|\varphi''(x_m)|)^{1/2}} \int_{-\infty}^{\infty} \exp\left(-\tau^2/2 + \sum_{n=3}^{\infty} \left(\frac{\tau}{\sqrt{\varphi''}}\right)^n \frac{1}{k^{n/2-1}}\varphi^{(n)}(x)\right) d\tau \quad \text{(A23)}$$

For $k \to \infty$ the terms with $n \geq 3$ go to zero and the integral is Gaussian so we get

$$I = e^{k\varphi(x_m)}\left(\frac{2\pi}{|\varphi''(x_m)|k}\right)^{1/2} \quad \text{(A24)}$$

So

$$\frac{\ln I}{k} = \varphi(x_m) + \frac{1}{2k}\ln\left(\frac{2\pi}{k|\varphi''(x_m)|}\right) \quad \text{(A25)}$$

$$\lim_{k\to\infty} \frac{\ln I}{k} = \varphi(x_m) \quad \text{(A26)}$$

so

$$\lim_{k\to\infty} I = e^{k\varphi(x_m)} \quad \text{(A27)}$$

$\varphi(j)$, (A19), has a maximum for k large at $j_{\max} = kC$, so (using the above theorem) we get for $k \to \infty$,

$$f_{S|L}(s|l) = \frac{\beta l(\beta s l)^{CC_2 - 1} e^{-\beta s l}}{\Gamma(CC)_2} \quad \text{(A28)}$$

which is identical to (41).

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R. L. Bras, I. Rodriguez-Iturbe, and D. G. Tarboton, Ralph M. Parsons Laboratory, Department of Civil Engineering, Massachusetts Institute of Technology, Cambridge, MA 02139.

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