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# The Fractal Nature of River Networks

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Ever since Mandelbrot (1975, 1983) coined the term, there has been speculation that river networks are fractals. Here we report analyses done on river networks to determine their fractal structure. We find that the network as a whole, although composed of nearly linear members, is practically space filling with fractal dimension near 2. The empirical results are backed by a theoretical analysis based on long-standing hydrologic concepts describing the geometric similarity of river networks. These results advance our understanding of the geometry and composition of river networks.

## Introduction

*Davis* [1899, p. 495] writes:

> Although the river and hill-side waste do not resemble each other at first sight, they are only the extreme members of a continuous series, and when this generalization is appreciated, one may fairly extend the "river" all over its basin and up to its very divides. Ordinarily treated, the river is like the veins of a leaf; broadly viewed it is like the entire leaf.

We see that the idea of river networks being generalized or extended to cover the whole basin is not new. In this paper we give some substance to these ideas by expressing them quantitatively, using notions of scaling and fractal dimension that have been recently suggested [*Mandelbrot*, 1975, 1983]. Fractals provide a mathematical framework for treatment of irregular, seemingly complex shapes that display similar patterns over a range of scales. Many objects in nature possess a property called statistical self-similarity. This may be defined as invariance of the probability distributions describing the object's composition under simple geometric transformations or change of scale. We argue that river networks fall into this class of geometric objects and that the fractal dimension characterizing the self-similarity of river networks is close to 2. For precise mathematical definitions of the notion of fractal dimension see *Voss* [1986] or *Mandelbrot* [1985]. *Wheatcraft and Tyler* [1988] give a concise summary of the important ideas.

*Horton* [1945] found that natural channels, when ordered a certain way, have bifurcation and length ratios that are approximately constant. These describe the scaling properties of river networks and have been important in river basin hydrology [*Rodriguez-Iturbe and Valdes*, 1979] and other disciplines like biology, where tree patterns occur [*Horsfield*, 1980; *Macdonald*, 1983].

Indirect empirical evidence gathered by hydrologists and geomorphologists has fueled speculation on the possible fractal nature of river basins. For example, *Gray* [1961] has reported relationships between mainstream length and basin area of the form

$$L = 1.4A^{0.568} \tag{1}$$

Based on (1), *Mandelbrot* [1983] has speculated that rivers are fractals with fractal dimension $D = 1.2$ (Mandelbrot took the exponent in (1) as 0.6; a better estimate is $D = 2 \times 0.568 \simeq 1.1$). This applies to individual rivers, rather than networks as a whole. *Mandelbrot* [1983] describes some fractal geometric patterns that resemble river networks where the fractal dimension of individual lines is 1.1, but the complete network pattern is space filling with $D = 2$. He suggests that these patterns are models of river networks.

The recent availability of Digital Elevation Models (DEM) of river basins (from the U.S. Geological Survey) and the continuously increasing power of computers permit a more careful study of the fractal dimension issue. Channels with varying resolution, or detail, can be defined from the DEM. *Band* [1986] discusses methods for obtaining channel networks from DEMs. The technique we use, suggested by L. E. Band (personal communication, 1987), is based on work of *O'Callaghan and Mark* [1984]. DEM data is supplied on a rectangular grid with each point representing the elevation of a 30 m by 30 m area or pixel. A drainage direction is assigned from each pixel toward one of its eight neighbors, based on the steepest slope. This effectively defines a drainage path or flow field. The number of pixels draining through each pixel is then counted to give the accumulated area that drains into each pixel. Channel networks are then defined as those pixels that have accumulated drainage area greater than a threshold support area. Decreasing the support area results in a finer network of channels and is tantamount to increasing the resolution used to study the basin. The limit to this refinement is the 30 m by 30 m pixel size of the U.S. Geological Survey's DEMs. Figure 1 shows networks in a river basin defined with varying support area. From the figure we can pose two questions: (1) is the process fractal? (implying that finer structures are statistically indistinguishable from grosser representations and hence there is no dominating scale) and (2) if so, what is the fractal dimension? The fact that by definition a river network drains its entire basin suggests the hypothesis that the fractal dimension of river networks (if it exists) should approach 2 (space filling). In the next section we show results that indicate that this is indeed the case. We then show a theoretical relationship between the scaling laws of network composition [*Horton*, 1945] and fractal dimension.

### Empirical Evidence

In this section we estimate fractal dimension of a river network using the Richardson method, functional box counting, and the distribution of stream length exceedances.

*Mandelbrot* [1983] describes how the following technique due to *Richardson* [1961] (in the study of the length of coastlines) provides an estimate of fractal dimension. Here it is applied to streams extracted from a DEM. Consider a line



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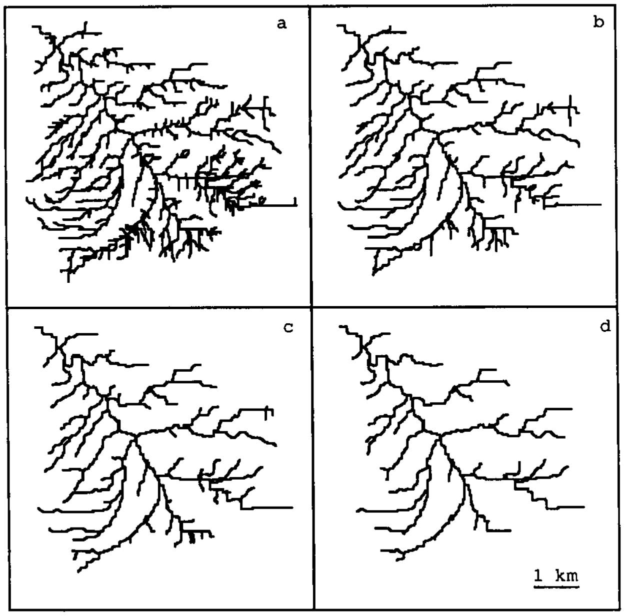


Fig. 1. W15 River network on a DEM square located in Walnut Gulch, Arizona. Channels defined with varying support areas (*a*) 20 pixels, (*b*) 50 pixels, (*c*) 100 pixels, (*d*) 200 pixels.

shape (e.g., coastline or stream) and measure its length, by stepping along it with dividers or a ruler of length $r$. The length is approximately $L = Nr$, where $N$ is the number of divider steps. By taking $r \rightarrow 0$ we should be able to converge to the exact length, i.e.,

$$L = \lim_{r \rightarrow 0} Nr \tag{2}$$

or

$$N \simeq Lr^{-1} \tag{3}$$

as $r$ approaches zero. However, Richardson found that the above limit often did not converge. The problem is in the implied exponent on $r$ in (2) being one. By allowing the exponent to be a fraction $D$, a measure $F$, independent of $r$, is obtained

$$F = Nr^D = \text{const} \tag{4}$$

with $D > 1$. Mandelbrot called this $D$ the fractal dimension. The above implies

$$N \sim r^{-D} \tag{5}$$

or equivalently,

$$L \sim r^{1-D} \tag{6}$$

Equation (6) indicates that on a log-log plot of length versus ruler size the fractal dimension is one minus the slope.

In applying the stepping procedure to river networks, rules are required to deal with bifurcations. Here we measured the length of each Strahler stream (defined according to *Strahler*'s [1952] network-ordering convention) separately. At the end of streams there is, generally, a leftover piece of stream shorter than $r$. If the distance from the last stepping point to the end was greater than $1/2\ r$, it was counted in $N$; otherwise, it was not included in the length. Figure 2 gives results for several different networks. The Souhegan is a 440-km$^2$ basin in southern New Hampshire that was digitized by hand from 1:24,000 U.S. Geological Survey maps. The Hubbard (area 75 km$^2$) in Conneticut, and W15 (area 23 km$^2$) in Walnut Gulch, Arizona, were extracted from DEMs. The eight networks in the sum consisted of two hand-digitized networks in New Hampshire, and six networks obtained from three DEM basins with support area's of 50 and 20 pixels. The DEM basins used were the Hubbard, W15, and W7 (also Walnut Gulch, Arizona). The pattern for all of these is the same, a gently sloping line with slope about 0.05 for small ruler lengths, followed by an abrupt change to slope of about one for large ruler lengths. This clearly indicates two distinct regions of scaling. The first, with $D \simeq 1.05$, is due to the sinuosity of individual rivers and corresponds to the scaling implied by (1). The second, with $D$ near 2, is due to the branching characteristic of networks. More precisely, it is due to streams shorter than $1/2\ r$ not being counted at all, reflecting the fact that at coarse resolution we see fewer streams.

Another technique to estimate the fractal dimension of channel networks is functional box counting as described by *Lovejoy et al.* [1987]. This works on a set of points (in this case $30 \times 30$ m pixels on a stream) embedded in a $d$-dimensional (here $d = 2$) space. Cover the space with a mesh of $d$-dimensional cubes of size $r^d$. Let $N(r)$ be the number of cubes that contain elements of the set considered. A relationship of the form

$$N(r) \sim r^{-D} \tag{7}$$

indicated by a straight line on a log-log plot gives $D$. Note that this is based on a definition of fractal dimension given by *Hentschel and Procaccia* [1983] as

$$D = -\lim_{r \rightarrow 0} \lim_{m \rightarrow \infty} \frac{\log N(r)}{\log r} \tag{8}$$

where $m$ is the number of points in the set. Results are given in Figure 3 for a river network defined for two different channel support areas. We see that there are basically two asymptotic slopes, a slope close to $-1$ for small box size, implying that at

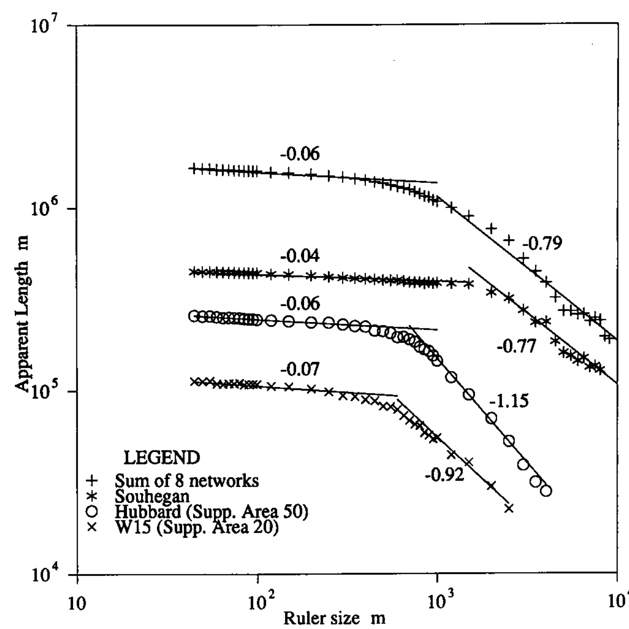


Fig. 2. Richardson method results for typical river networks. The numbers give slope of the fitted straight lines.

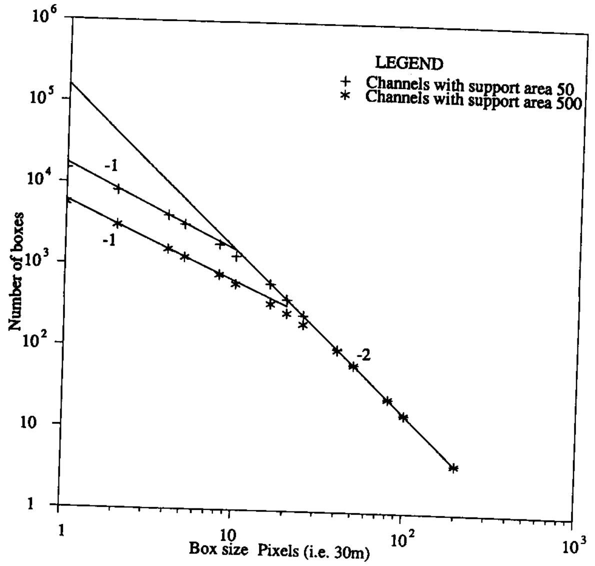


Fig. 3. Functional box counting results from square in Walnut Gulch, Arizona. The numbers give slope of the fitted straight lines.

scales small relative to the resolution of the map, channels have dimensions close to that of the line. At the large box-size end of the scale the slope is $-2$, indicating that practically all boxes are intersected by a channel. At this scale the network is space filling with $D = 2$. Note that the more detailed the network (smaller support area) the smaller the scale above which the network is space filling.

As previously mentioned, the region with slopes near $-1$ in Figure 2 is due to short streams being excluded as $r$ increases. In the region with slope of $-1$, (4) can be interpreted as giving the number of streams with length greater than $r$ proportional to $r^{-D}$. *Mandelbrot* [1983] notes that the probabilistic counterpart to this is a hyperbolic distribution:

$$\text{Prob}[\text{length} > l] \sim l^{-D} \tag{9}$$

where $D$ is again the fractal dimension and $l$ refers to stream length. Hyperbolic distributions have the desired property that they are self-similar.

Figures 4 and 5 give the exceedance probability of stream length aggregated from several river basins. Points were plotted using

$$P = m/(n+1) \tag{10}$$

where $m$ is the ranking from longest to shortest stream length, and $n$ is the number of streams in the sample. The figures indicates a hyperbolic tail with $D \simeq 2$. Figure 4 uses geometric length, defined as the straight-line distance between endpoints of a stream. Figure 5 uses length measured along the stream, naturally limited by the resolution of the map or DEM from which the network is obtained. The slight difference in slope between these figures may be due to length along the stream itself being a fractal measure with dimension $D$ slightly in excess of 1. As an example, suppose we have, from (9), fitted to Figure 5,

$$\text{Prob}[\text{length} > l] \sim l^{-\lambda} \tag{11}$$

Now if $l$ is itself a fractal with dimension $D_l$, we get, from (4),

$$l \sim r^{D_l} \tag{12}$$

where $r$ is a linear ($D = 1$) measure or length scale. Combining, we get

$$\text{Prob}[\text{length} > l] \sim r^{-\lambda D_l} \tag{13}$$

Thus the fractal dimension of the whole network is $D = \lambda D_l$. The result $D = 2$ is therefore consistent with slope $\lambda = 1.8$ seen in Figure 5 and $D_l = 1.1$ as suggested by (1) and the flatter

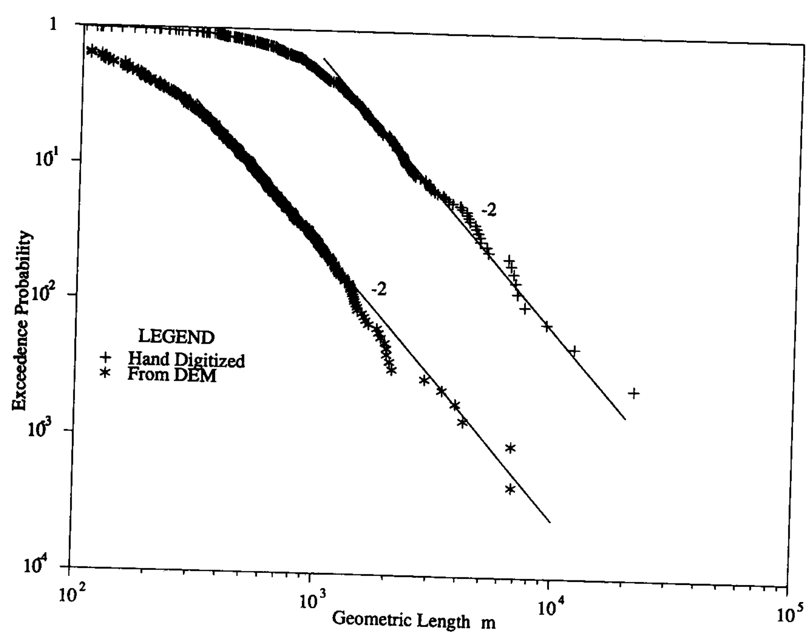


Fig. 4. Geometric stream length exceedence probability. The DEM data is based on 2178 streams from three networks with support area of 20 pixels. The digitized data is based on two networks with 409 streams digitized by hand from 1:24,000 maps.

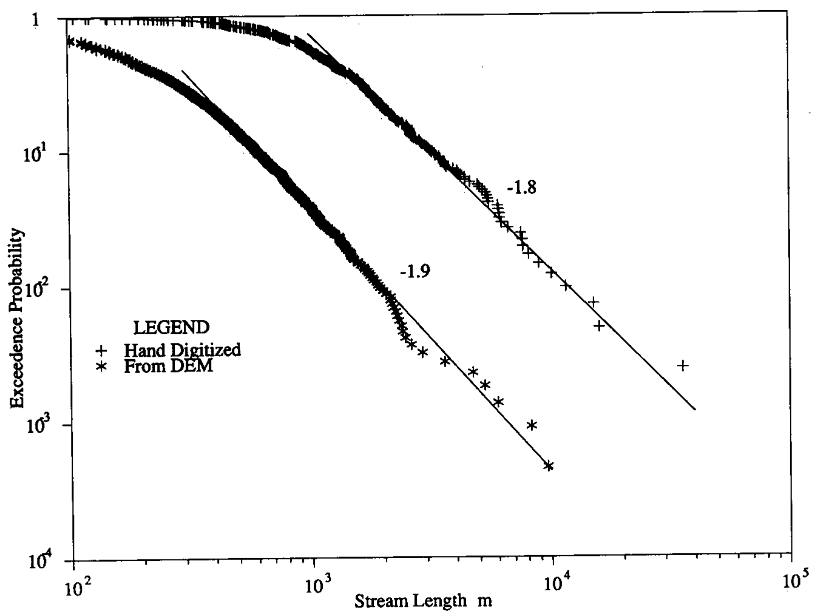


Fig. 5. Stream length (along stream) exceedance probability. The DEM data is based on 2178 streams from three networks with support area of 20 pixels. The digitized data is based on two networks with 409 streams digitized by hand from 1:24,000 maps.

slopes of Figure 2. We interpret these figures as strong evidence that the network is space filling with $D = 2$.

*Theory and Explanantion*

In this section we will show how fractal dimension is related to *Horton* [1945] laws of network composition and hence put the previous empirical evidence in the framework of classical fluvial geomorphology.

Horton's laws, in particular, the length and bifurcation ratios, are usually stated in terms of *Strahler*'s [1952] ordering scheme. Source streams are of order one. When two first-order streams join, they become second order, and, in general, when two streams of equal order merge, a stream one order higher is formed. When low- and high-order streams join, the continuing stream retains the order of the higher-order stream.

The set of empirical laws collectively referred to as Horton's laws include:

Bifurcation

$$R_b = \frac{N_{\omega-1}}{N_\omega} \tag{14}$$

Length

$$R_l = \frac{L_\omega}{L_{\omega-1}} \tag{15}$$

where $N_\omega$ is the number of streams of order $\omega$, and $L_\omega$ is the mean length of stream of order $\omega$. $R_b$ and $R_l$ can be obtained from the slopes of the straight lines resulting from plots of log $N_\omega$ and $L_\omega$ versus order $\omega$.

The above are geometric-scaling relationships, since they hold no matter at what order or resolution we view the network. If we regard a channel network as paths where water flows, it is possible to imagine [after *Davis*, 1899], with higher and higher resolution, getting lower and lower orders of streams until we are literally looking at flows among the grass roots. Viewed this way, the limiting channel network is a fractal, with properties governed by $R_b$ and $R_l$.

Based on Horton's laws, *LaBarbera and Rosso* [1987] report that the fractal dimension of river networks should be

$$D = \max\left(\frac{\log R_b}{\log R_l}, 1\right) \tag{16}$$

A derivation of the above given in Appendix A requires that Horton's bifurcation and length ratios hold exactly at all scales in the network. Then the total length of streams in the network is the sum of a geometric series with multiplier $R_b/R_l$. The result is obtained by considering the limit of this series which converges for $R_b < R_l$, implying $D = 1$. However, when $R_b > R_l$, the series diverges, and we show in Appendix A that the total length of channel networks follows

$$L \sim s^{1-(\log R_b/\log R_l)} \tag{17}$$

where the resolution of observation of the network $s$ is taken as the length's of first-order streams. By analogy to (6),

$$D = \frac{\log R_b}{\log R_l} \tag{18}$$

In Appendix B we show that Horton's laws can be used to give the stream length exceedance probability distribution as

$$\text{Prob(length} > l) \sim l^{-(\log R_b/\log R_l)} \tag{19}$$

Comparing this with (9), we again get (18).

Table 1 gives Horton ratios for several river basins, indicating that $D$ estimated by (18) are scattered around 2. The results presented seem to indicate that river networks are scaling and have fractal dimension near 2.

## CONCLUSIONS

We have shown that river networks can be viewed as fractal, and estimates of the fractal dimension using three different

TABLE 1. Network Geometry Data for Several River Networks

| Basin | Magnitude | $R_b$ | $R_l$ | $\frac{\log R_b}{\log R_l}$ |
|---|---|---|---|---|
| Souhegan, New Hampshire | 177 | 3.5 | 2.0 | 1.8 |
| Squanacook, New Hampshire | 133 | 3.5 | 1.7 | 2.5 |
| W15 Support A 20 | 329 | 4.2 | 2.1 | 1.9 |
| W15 Support A 50 | 107 | 3.3 | 1.6 | 2.4 |
| Hubbard Support A 20 | 1217 | 4.1 | 2.1 | 1.9 |
| Hubbard Support A 50 | 486 | 4.7 | 2.3 | 1.8 |
| Youghiogheny River, Maryland* | 1798 | 4.6 | 2.2 | 1.9 |
| Daddys Creek, Tennessee | 1181 | 4.1 | 2.2 | 1.8 |
| Allegheny River, Pennsylvania | 5966 | 4.5 | 2.4 | 1.7 |

*Based on data from *Morisawa* [1962].

techniques all tend to indicate that the fractal dimension $D$ is 2. This is consistent with the fact that rivers drain the entire catchment basin and are thus space filling. We have shown how the fractal dimension of rivers is related to *Horton*'s [1945] classical laws of network composition. If the result $D = 2$ is accepted, it implies $R_b = R_l^2$, thus providing a fundamental link between Horton's ratios. This appears to be borne out in practice within the scatter in estimation of $R_b$ and $R_l$. $R_b$ and $R_l$ appear to both describe the same scaling property evident in river basins. It is also worth noting that the random topology model of *Shreve* [1967] gave average values $R_b = 4$ and $R_l = 2$. It is therefore consistent with $D = 2$. The view of river networks as fractal with $D = 2$ therefore provides a description of the scaling of river networks that is consistent with classical fluvial geomorphology and the popular random topology model. The implications of this result to quantitative hydrology could be significant.

## APPENDIX A

Let a network of order $\Omega$ have main stream length $L_\Omega$. Then, using the Horton's length ratio, the mean length of a stream of order $\omega (\omega < \Omega)$ is $L_\Omega/(R_l)^{\Omega-\omega}$. By Horton's bifurcation law there are $R_b^{\Omega-\omega}$ of these streams, so that the total length of stream of order $\omega$ is $L_\Omega(R_b/R_l)^{\Omega-\omega}$. Adding over all $\omega$ to get the length of the whole network $L$, we get the geometric series

$$L = \sum_{\omega=1}^{\Omega} L_\Omega(R_b/R_l)^{\Omega-\omega}$$

$$L = L_\Omega[1 - (R_b/R_l)^\Omega]/[1 - (R_b/R_l)] \tag{A1}$$

*Strahler* [1964] gives this result. If $R_b/R_l < 1$, the series converges to a finite $L$ as $\Omega$ approaches infinity and we have $D = 1$. Remember that this is a limit process where $L_\Omega$ is held constant and $\Omega$ increases as the resolution is refined. However, if $R_b/R_l \geq 1$ as is most often the case in river channel networks, the series diverges and for large $\Omega$ we get

$$L \sim (R_b/R_l)^{\Omega-1} \tag{A2}$$

Now the first-order streams have average length

$$s = (1/R_l)^{\Omega-1} \tag{A3}$$

This is ~~integrated~~ interpreted as the resolution used to measure the length of the network. From (A3), write

$$\Omega - 1 = -(\log s/\log R_l) \tag{A4}$$

which in (A2) gives

$$L \sim s^{1-(\log R_b/\log R_l)} \tag{A5}$$

## APPENDIX B

In a river network with Horton's bifurcation and length ratio laws holding exactly, we have from Appendix A, that there are $R_b^{\Omega-\omega}$ order $\omega$ streams of length $L_\Omega/R_l^{\Omega-\omega}$. So the total number of streams exceeding a length $l = L_\Omega/R_l^k$ is

$$\sum_{i=0}^{k} R_b^i = (R_b^{k+1} - 1)/(R_b - 1) \qquad k = \frac{\log (L_\Omega/l)}{\log R_l} \tag{B1}$$

If the total number of streams is $N_T$, we can write

$$\text{Prob(length} \geq l) = [(R_b^{k+1} - 1)/(R_b - 1)]/N_T \tag{B2}$$

For $k$ large so that $R_b^{k+1}$ dominates 1 this becomes

$$\text{Prob(length} \geq l) \sim l^{-(\log R_b/\log R_l)} \tag{B3}$$

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