Fractals, Vol. 1, No. 3 (1993) 606–616

UNDERSTANDING COMPLEXITY IN THE STRUCTURE OF RAINFALL

BALAJI RAJAGOPALAN and DAVID G. TARBOTON
Utah Water Research Laboratory, Dept. of Civil Engg.
Utah State University, Logan, Utah – 84322-8200, USA

Abstract

This paper aims at understanding the structure of high resolution storm rainfall rates. Scale invariance property has been observed from the distribution function. Departures from simple scaling (i.e., multiscaling) have been noted, suggesting a cascade phenomenon. Close agreement of theoretical and estimated singularity spectra indicates the plausibility of modeling rainfall process by random cascades. Connections between multiscaling and the singularity spectrum have been identified.

1. INTRODUCTION

Study of the rainfall process and its scaling properties, both spatial and temporal is of great importance in hydrometrology. The traditional approach is to analyze the statistical dependence structure of the precipitation process by stochastic modeling, which has resulted in a large number of stochastic models. The statistical dependence structure has been shown by Rodriguez-Iturbe et al.[1] to be closely tied to the scales at which descriptions are sought. Consequently, the parameter estimation of any stochastic model used for modeling the rainfall process is also linked to the scales of the measured rainfall. This poses problems for the definition and identification of universal models for the rainfall occurrence process. In order to be able to understand the rainfall process over a wide range of scales, scale considerations are very important while modeling the rainfall process in time/space.

Phone: 801-750-3172
Fax: 801-750-3663
Email: rbala@pub.uwrl.usu.edu

Lovejoy and Schertzer[2] introduced the ideas of scale invariance and fractals into hydrologic rainfall modeling through evidence that rainfall may be scaling. Schertzer and Lovejoy[3] note that the rainfall process possesses no characteristic scale, i.e., the process can be said to be scale invariant. Rodriguez-Iturbe et al.[4] and Sharifi et al.[5] have studied storm rainfall in detail and have found evidence of deterministic chaos in the process, indicating that the underlying dynamics is deterministic. These serve as motivating factors for exploring the scaling properties to better understand the structure present in the rainfall process.

We present the analysis of storm rainfall rates, towards possible modeling by a simple random cascade. Simple and multiscaling properties have been studied. Singularity spectra have been estimated using data driven methods and connections have been established with multiscaling. Theoretical singularity spectrum from a multiplicative random cascade has been fit to the estimated spectrum. Close agreement between both the spectra provides justification for modeling the rainfall process with a random cascade. Data sets are first described, analysis of scale invariance is next presented. Probings into the scaling properties of the moments are then discussed. Singularity spectrum analysis and their concurrence with moment scaling are next presented. Discussion of the results concludes the paper.

2. DATA DESCRIPTION

Two very high resolution rainfall rate data sets were used in this study. Data from a storm in Indiana (USA) was collected using a distrometer (a LASER instrument) by Dr. Huang. Rainfall amounts each ten seconds are determined from drop sizes and counts of the number of drops in each size class over a 50 cm^2 area. In this paper the storm of October 9, 1990 was studied. Figure 2(a) shows the data. The other data set analyzed was the Boston storm of October 25, 1980. Rainfall depth was accumulated every fifteen seconds by a high resolution tipping bucket rain gauge located at MIT (USA). Figure 1(a) shows this data. The Boston data was the same data as used by Rodriguez-Iturbe et al.[4] in their search for deterministic chaos. They analyzed the correlation dimension of various embedding dimensions and concluded that this data suggested the existence of a low dimensional attractor.

3. EVIDENCE FOR SCALING

Rodriguez-Iturbe[6] note the intermittency and strong dependence of statistical moments on a few data values for the Boston storm data. Similar behavior was seen in the Indiana storm data. This is evidence that scale dependence may be important.

Lovejoy and Schertzer[2] studied scaling in rainfall rate using radar measured data by looking at the tails of the exceedence probability distribution function. They note that

$$\mathrm{Prob}(|\Delta R| > \Delta r) \sim \Delta r^{-\beta} \tag{1}$$

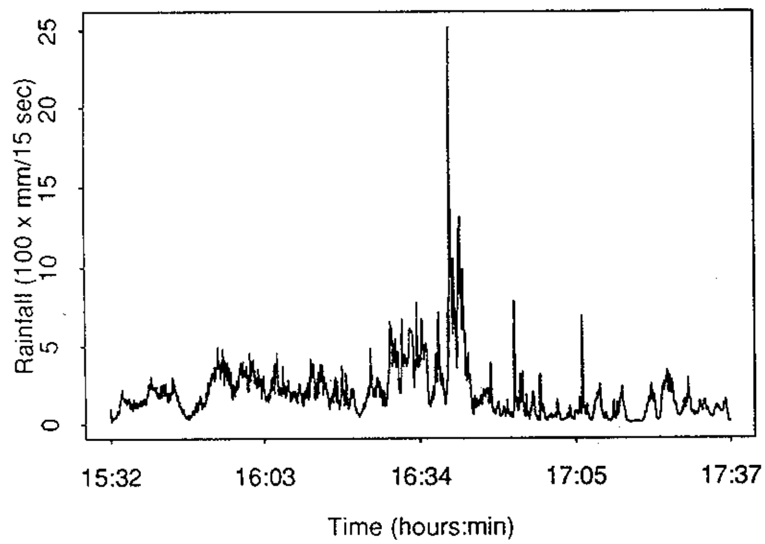
In other words, the probability that the change of rainfall flux ΔR between two instants separated by a time ΔT is larger than any value Δr, and is proportional to $\Delta r^{-\beta}$. Equation (1) is a power-law relation and plots as a straight line in double logarithmic paper, with β the slope of the line.

We did a similar analysis. Here $\Delta R(\Delta T)(= R(t_2) - R(t_1))$ is the difference between successive depths of rainfall that accumulates in interval $\Delta T(= t_2 - t_1)$. ΔT is called the time of aggregation. Four times of aggregation were chosen for the present study. For the

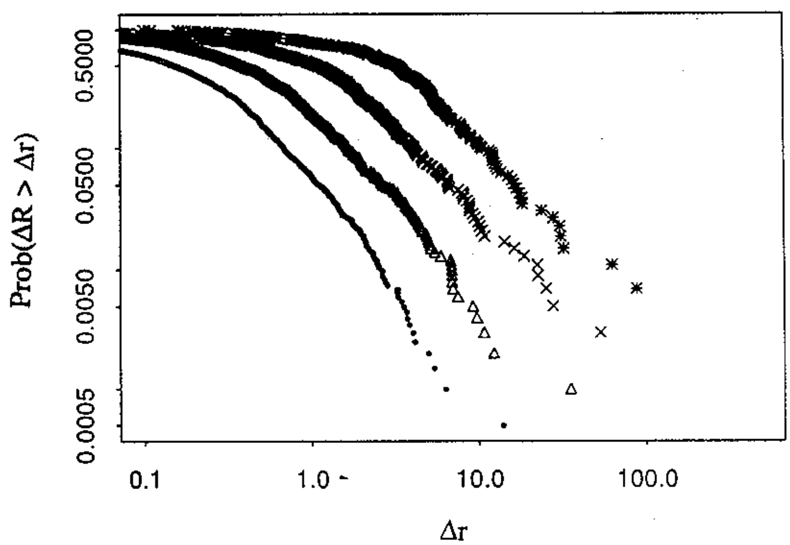
Boston storm the ΔTs are 15 sec., 30 sec., 60 sec. and 120 sec., while for the Indiana storm they are 10 sec., 20 sec., 40 sec. and 80 sec. Figures 1(b) and 2(b) are the plots of $\text{Prob}(|\Delta R| > \Delta r)$ vs. Δr on a logarithmic scale for various ΔTs for the Boston and Indiana storm, respectively. The Weibull plotting position formula[7] was used. The striking feature of the plots is that, doubling ΔT does not change the separation between the lines and the tails are practically parallel. This is suggestive of self-similarity.

Self-similarity and scaling in space-time rainfall is discussed bys Waymire and Gupta.[8] They define a process $X(t)$ as simple scaling if upon rescaling $t \to \lambda t$, and defining:

$$X_\lambda(t) = X(\lambda t), \quad \lambda > 0 \tag{2}$$

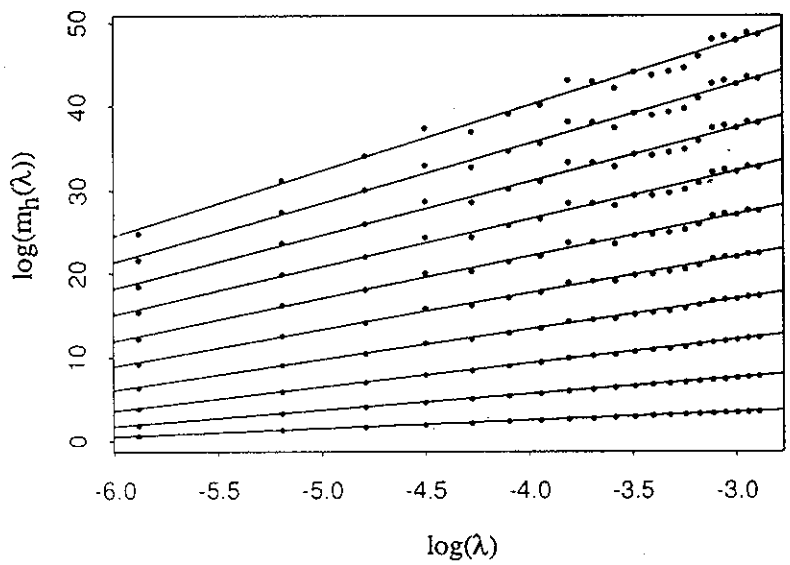


(a)

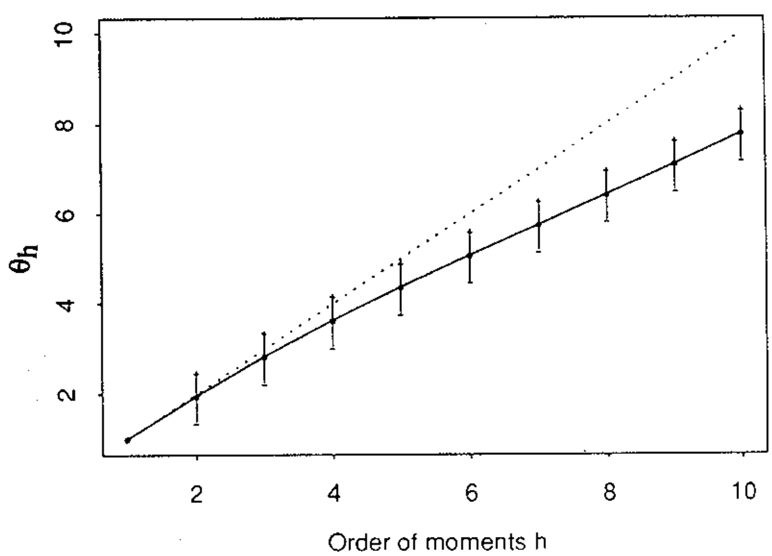


(b)

Figs. 1(a)–1(d) (a) Time series plot of the October 25, 1980 storm in Boston. (b) Probability distributions of rainfall fluctuations for the October 25, 1980, storm in Boston, plotted on log-log scale.



(c)



(d)

Fig. 1 (*Continued*)
(c) $\log(m_h(\lambda))$ vs. $\log(\lambda)$. Plotted for first ten moments ($h = 1, \ldots, 10$), in ascending order from the bottom. (d) Departures from simple scaling in the growth of slopes with respect to order of the moments. The dotted line indicates simple scaling, solid line is the estimate from the data. The vertical bars are 95% confidence bands, estimated as two times the standard error of the regression slope from Fig. 1(c).

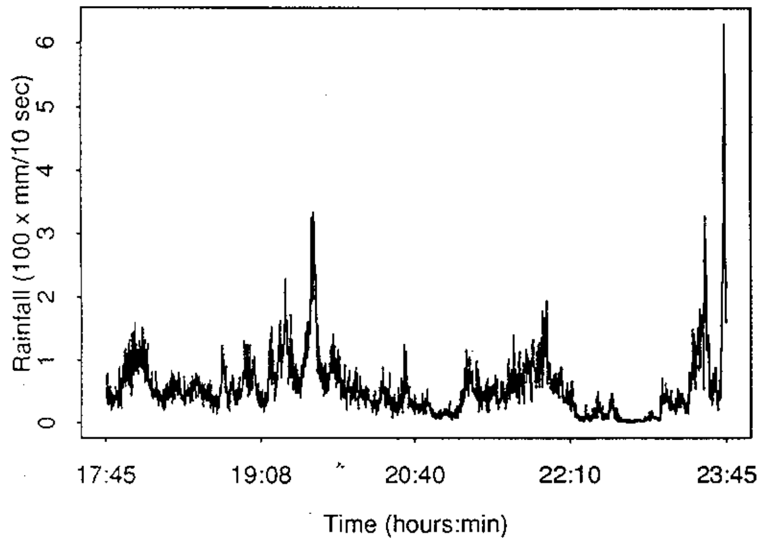
the following relation holds,

$$X_\lambda(t) \stackrel{d}{=} \lambda^\theta X(t) \tag{3}$$

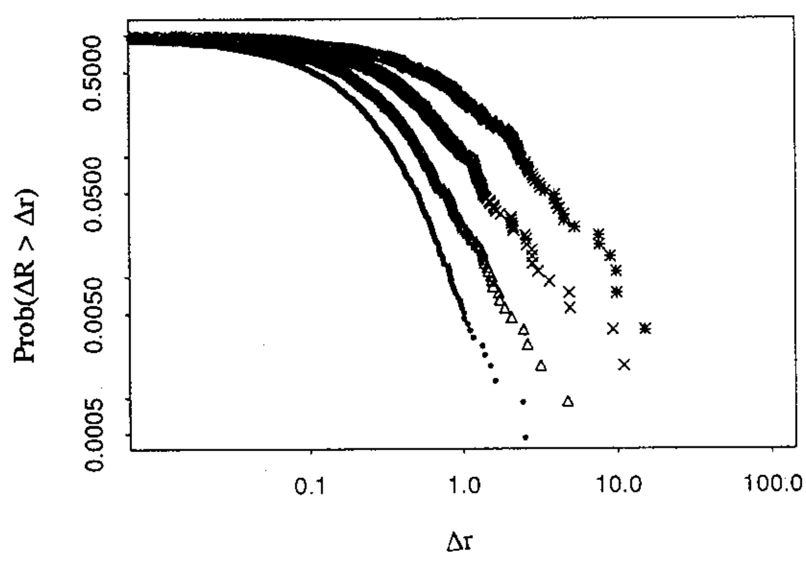
θ can be either positive or negative and is called the scaling exponent, $\stackrel{d}{=}$ denotes identity in statistical distribution.

Simple scaling implies that there is symmetry across scales in the irregularities of the process, reflecting an underlying structural self-similarity which can be characterized by *fractal dimension* D.[9] The exponent β and the fractal dimension D are related as[6]:

$$D = 2 - 1/\beta \tag{4}$$

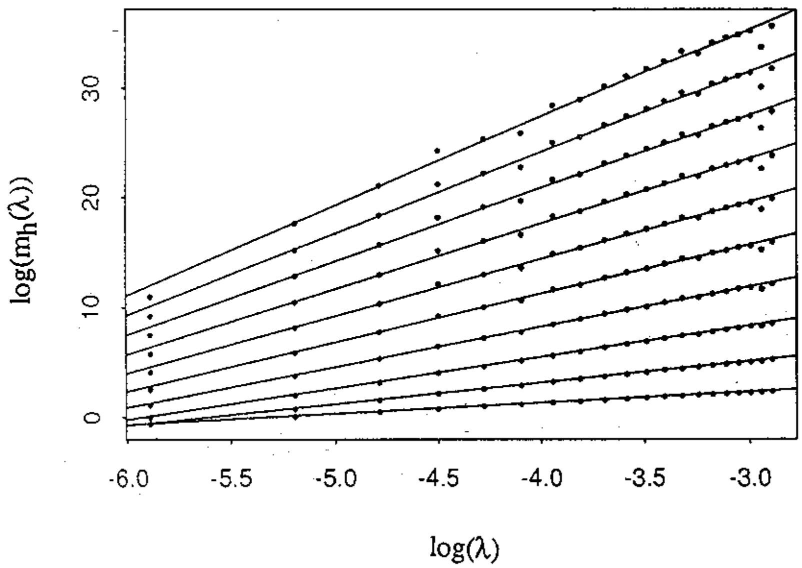


(a)

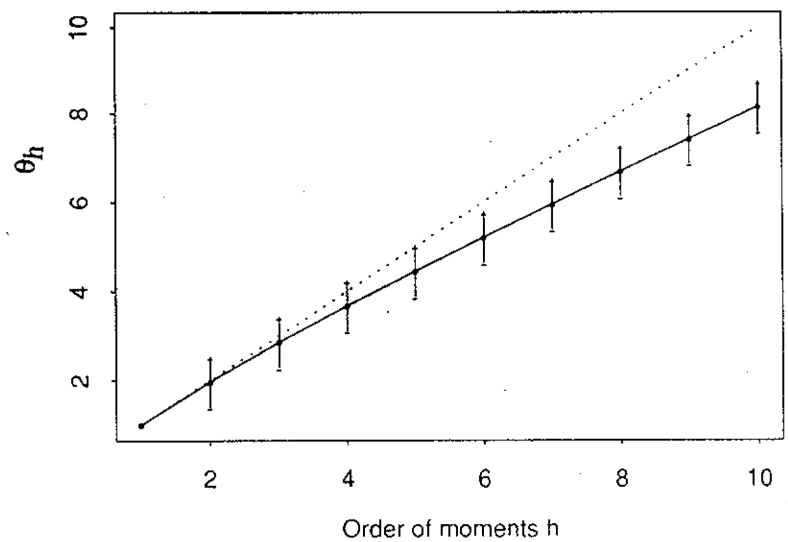


(b)

Figs. 2(a)–2(d) (a) Time series plot of the October 9, 1990 storm in Indiana. (b) Probability distributions of rainfall fluctuations for the October 9, 1990, storm in Indiana, plotted on a log-log scale.



(c)



(d)

Fig. 2 (*Continued*)
(c) $\log(m_h(\lambda))$ vs. $\log(\lambda)$. Plotted for first ten moments ($h = 1, \ldots, 10$), in ascending order from the bottom. **(d)** Departures from simple scaling in the growth of slopes with respect to order of the moments. The dotted line indicates simple scaling, solid line is the estimate from the data. The vertical bars are 95% confidence bands, estimated as for Fig. 1(d).

4. SCALING OF MOMENTS

Properties of random variables are frequently studied through the analysis of moments. Gupta and Waymire[8] have studied the moment scaling of spatial rainfall process and have developed a class of multiplicative processes in light of their observations. Here, we investigated the moment scaling of temporal rainfall process and our results support their findings.

If the process $X_\lambda(t)$ has finite moments $E[X_\lambda^h(t)]$ of order h and if it is simple scaling, then we can write:

$$E[X_\lambda^h(t)] = \lambda^{h\theta} E[X^h(t)] \tag{5}$$

By taking log on both sides we get,

$$\ln\left(E[X_\lambda^h(t)]\right) = h\theta \log(\lambda) + \log\left(E[X^h(t)]\right) \tag{6}$$

Denoting, $m_h(\lambda) = E[X_\theta^h(t)]$, and $m_h(1) = E[X^h(t)]$, we can write Eq. (6) as:

$$\ln(m_h(\lambda)) = h\theta \ln(\lambda) + \ln(m_h(1)) \tag{7}$$

This log-linear scaling of the moments is of the general form:

$$\ln(m_h(\lambda)) = a_h + \theta_h \ln(\lambda) \tag{8}$$

With $\theta_h = h\theta$, the scaling exponent corresponding to moment h. Scaling only of the moments (as opposed to pdf) is called wide sense scaling by Gupta and Waymire.[8] One can imagine (and observe) that moment scaling is possible with θ_h nonlinear in h. This form of scaling is more general than simple scaling and is called multiscaling. The moment scaling exponent θ_h is related to the mass exponent $\tau(q)$ and singularity spectrum $f(\alpha)$ more commonly used to characterize multiscaling. These relationships are discussed below.

First we analyzed the moment scaling properties of the storm data. $m_h(\lambda)$ vs. λ is plotted on a log-log plot for each order of moment h. In this case, the first ten moments were evaluated. Figures 1(c) and 2(c) indicate this. The slope θ_h for each h is estimated via regression. Figures 1(d) and 2(d) are the plots of θ_h vs. h. The dotted line is the linearity of slope change assuming simple scaling ($\theta_h = h\theta$), while the solid line is the one obtained from the data analysis as described above.

From Figs. 1(c) and 2(c), it can be seen that the log-log linearity of moment scaling is preserved. However from Figs. 1(d) and 2(d) the linearity of slope change displays concave departures from simple scaling. Gupta and Waymire[8] show that by virtue of concavity only scale magnification ($\lambda < 1$) is possible. They[8] suggest this is a consequence of "cascading down" of some large-scale flux to successively smaller scales. This feature is similar to the cascading of the energy flux in fully developed turbulent flows. This suggests that the rainfall process can be modeled using cascade processes. This is further investigated in the next section.

5. MULTIFRACTALS

Physical systems that exhibit chaotic behavior are common in nature. Such systems lose information exponentially fast; as a result it is possible to follow and predict their motion in any detail only for short time scales. Chhabra et al.[10] argue that, to describe their long term dynamical behavior, one must resort to suitable statistical descriptions and one such description is the multifractal formalism. They say that the multifractal formalism comes from the fact that highly nonuniform probability distributions arising from the nonuniformity of the system often possess rich **scaling properties** including that of self-similarity.

In particular, if we cover the support of the measure with boxes of size λ and define $P_i(\lambda)$ to be the fraction of integrated measure in the ith box, then we can define an exponent (singularity strength) α_i by:

$$P_i(\lambda) \sim \lambda^{\alpha_i} \tag{9}$$

If we count the number of boxes $N(\alpha)$ where the fraction P_i has singularity strength between α and $\alpha + d\alpha$, then $f(\alpha)$ can be loosely defined as the fractal dimension of the set of boxes with singularity strength α by,

$$N(\alpha) \sim \lambda^{-f(\alpha)} \tag{10}$$

This leads to the notion that a fractal measure can be comprised of interwoven sets of singularities of strength α, each characterized by its own fractal dimension $f(\alpha)$. Such measures are termed multifractals.[11] The plot of α vs. $f(\alpha)$ is called the singularity spectrum.

The scaling exponent for the qth moments of the measure, provides an alternative description of the singular measure. Using the same notations as above, series of exponents parametrized by q can be defined as:

$$\sum_i P_i^q(\lambda) \sim \lambda^{-(q-1)D_q} = \lambda^{-\tau(q)} \tag{11}$$

D_q is called the generalized dimension; these characterize the nonuniformity of the measure: positive q's accentuate the denser regions and negative q's accentuate the rare ones. For special values of q, D_q can be realized as the dimension of a special set. which supports a particular part of the measure. For example, $D_{q=0}$ is the dimension of support of the measure, which is also the maximum of the singularity spectrum. $(q-1)D_q$ is defined as $\tau(q)$, also called the mass exponent. $f(\alpha)$, α, $\tau(q)$, q are related through a Legendre transformation. For details, the reader is referred to Chhabra et al.,[10] Feder,[12] Chhabra and Jensen.[13] The singularity spectrum provides a precise and intuitive description of the multifractal measure in terms of interwoven sets, with differing singularity strengths α, whose fractal dimension is $f(\alpha)$.

We next establish connections between multiscaling of moments described in the previous section and multifractals discussed above. It can be seen from Eq. (8) that:

$$m_h(\lambda) \sim \lambda^{\theta_h} \tag{12}$$

$m_h(\lambda)$ is the moment of order h at aggregation scale λ. Here h plays the same role as q. Now recognize that the fraction of rainfall in an interval i with length λ is $P_i(\lambda) = R_i(\lambda)/T$ where T is the total storm rainfall amount. Also recognize that the number of intervals of length λ used in the calculated of moments $N(\lambda) \sim \lambda^{-1}$. Therefore,

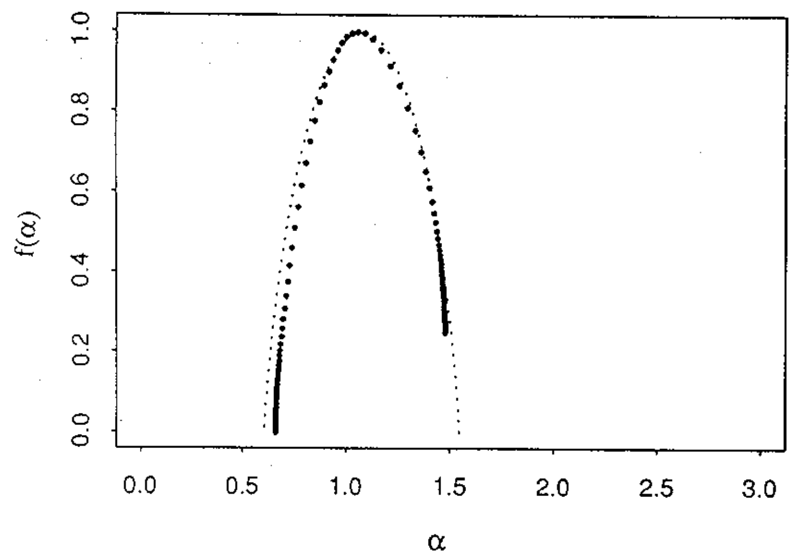
$$m_h(\lambda) = \frac{1}{N(\lambda)} \sum_i R_i^h(\lambda) = \frac{1}{N(\lambda)T} \sum_i P_i^h(\lambda) \sim \lambda^{1-\tau(q)} \tag{13}$$

Comparing to Eq. (12), it follows that

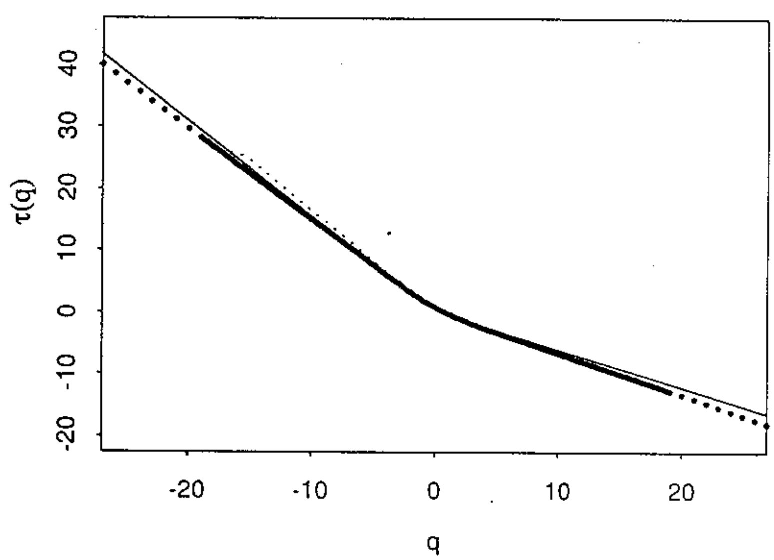
$$\theta_h = 1 - \tau(q) \tag{14}$$

Remember here that $h = q$. The $f(\alpha)$ curved can be computed using the Legendre transform of the $\tau(q)$ curve, or directly using Chhabra and Jensen's[13] method. It was observed that the method did not work unless we used box sizes λ, that were factors of length of the data.

The $f(\alpha)$ spectrum estimated using the direct method of Chhabra and Jensen,[13] is shown in Figs. 3(a) and 4(a), plotted as points. As moment scaling analysis suggested a cascade phenomena, we estimated the singularity spectrum of the best fitting random cascade, using closed form equations for random binomial cascades provided in Halsey et al.[14] These are plotted as dotted lines in Figs. 3(a) and 4(a). The estimated and theoretical spectra are in reasonable agreement; however, one could choose more complicated

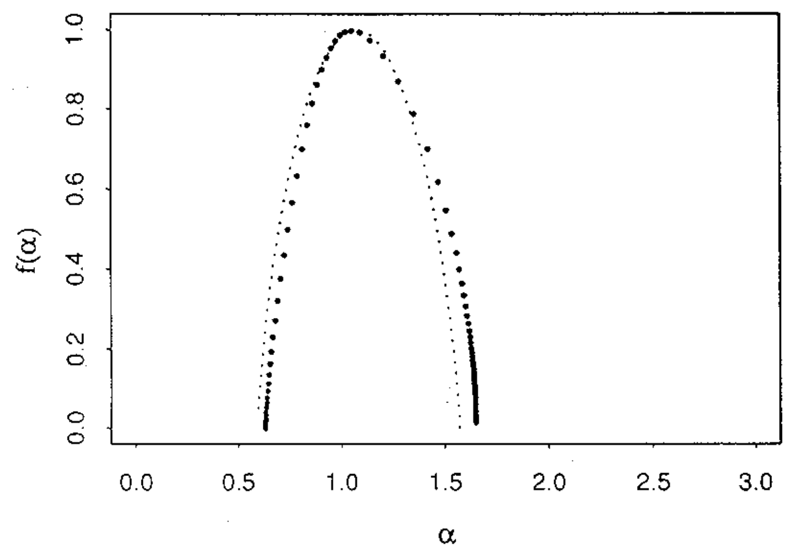


(a)

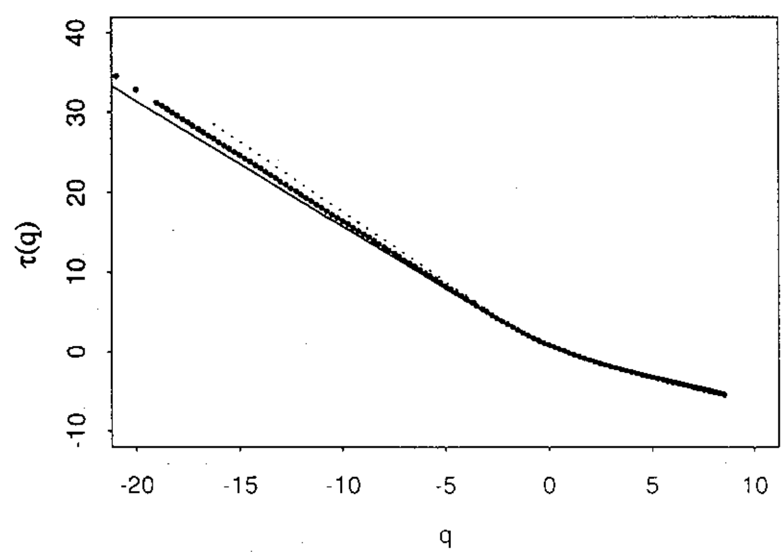


(b)

Fig. 3 (a) Singularity spectrum for the Boston storm. Points are estimated from the data using Chhabra and Jensen's method. The dotted line is the theoretical spectrum for a random binomial cascade with $p = 0.6575$. (b) $\tau(q)$ vs. q plot for the Boston storm. The points are estimated from the data. The dotted line is estimated from multiscaling [Eq. (14)]. The solid line is the curve for a random binomial cascade with $p = 0.6575$.



(a)



(b)

Fig. 4 (a) Singularity spectrum for the Indiana storm. Points are estimated from the data using Chhabra and Jensen's method. The dotted line is the theoretical spectrum for a random binomial cascade with $p = 0.6636$. (b) $\tau(q)$ vs. q plot for the Indiana storm. The points are estimated from the data. The dotted line is estimated from multiscaling [Eq. (14)]. The solid line is the curve for a random binomial cascade with $p = 0.6636$.

cascade models. Finally in Figs. 3(b) and 4(b), $\tau(q)$ curves calculated using the direct method of Chhabra and Jensen,[13] moment scaling [Eq. (14)] and from the fitted random cascade are compared. Closeness of all three lends confidence to the fact that the rainfall process can be modeled by cascade processes. This suggests that convections developed at larger time scales cascade down to smaller time scales. Meneveau and Sreenivasan[15] have developed a binomial cascade model (with $p = 0.7$) for fully developed turbulence. The plausibility of modeling the rainfall process by a random cascade with a similar parameter p, suggests ties between rainfall process and atmospheric turbulence.

After the appropriate cascade model is obtained, it could be used to simulate the rainfall process. Thus, we have a simple cascade model for the observed storms, which has the ability to reproduce the scaling properties of the process.

6. DISCUSSION

The analysis of storm data sets indicate that the seemingly irregular patterns exhibited by the storm rainfall, has a structure embedded in it that binds the process across a wide range of scales. The probability distributions show that the rainfall processes observed are scaling. The moment scaling analysis suggests multiscaling and as a result a cascading phenomenon. The establishment of connection between multiscaling of moments and the singularity spectrum, through $\tau(q)$ [Eq. (14)] implies that the two different analyses provide qualitatively, similar information. This lends confidence to the inferences. The close agreement of $\tau(q)$ curves estimated from all three approaches, and the singularity spectra both of the estimated and theoretical, suggest the presence of a cascading phenomena.

ACKNOWLEDGMENTS

Thanks are due to Dr. Chi-Hua Huang, (USDA-RAS-MWA, National Soil Erosion Research Laboratory, West Lafayette, IN) for providing the high resolution storm data sets of Indiana, and also Dr. Shatiq Islam (Univ. of Cincinatti) for providing the Boston storm data set. We are grateful for discussions with Tom Over.

REFERENCES

1. Rodriguez-Iturbe, V. K. Gupta and Ed Waymire, *Water Resources Research*, **18**(5), 1451 (1982).
2. S. Lovejoy and D. Schertzer, *Water Resources Research* **21**(8), 1233 (1985).
3. D. Schertzer and S. Lovejoy, *Journal of Geophysical Research* **92**(D8), 9693 (1987).
4. Rodriguez Iturbe, B. F. De Power, M. B. Sharifi and K. P. Georgakakos, *Water Resources Research* **25**(7), 1667 (1989).
5. M. B. Sharifi, K. P. Georgakakos and I. Rodriguez-Iturbe, *Journal of the Atmospheric Sciences* **47**(7), 888 (1990).
6. Rodriguez-Iturbe, *Transport Processes and the Hydrological Cycle*, (Venice, 1992), p. 240.
7. Chow, *Bulletin, Eng. Experiment Station*, (Univ. of Ill., Urbana), **414**, 26 (1953).
8. V. K. Gupta and Ed. Waymire, *Journal of Geophysical Research* **95**(D8), 1999 (1990).
9. B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman and Co., New York, 1982).
10. A. Chhabra, C. Meneveau, R. V. Jensen and K. R. Sreenivasan, *Phys. Rev.* **A40**(9), 5284 (1989).
11. Vicsek, *Fractal Growth Phenomena*, (World Scientific, Singapore, 1989).
12. Feder Jens, *Fractals* (Plenum Press, New York, 1989).
13. A. Chhabra and R. V. Jensen, *Phys. Rev. Lett.* **62**, 1327 (1989).
14. T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia and B. I. Shraiman, *Physical Review* **A33**(2), 1141 (1986).
15. C. Meneveau and K. R. Sreenivasan, *Phys. Rev. Lett.* **59**(13), 1424 (1987).