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Comment on "On the Fractal Dimension of Stream Networks" by Paolo La Barbera and Renzo Rosso

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La Barbera and Rosso [1989] give an expression for the fractal dimension of river networks in terms of Horton's ratios. This yields estimates of fractal dimension between 1.5 and 2 with an average between 1.6 and 1.7. The derivation of the expression relied on an implicit assumption that individual stream elements, in particular first-order streams, were linear measures with fractal dimension 1, despite the recognition that this is not the case. Here we show how the result of *La Barbera and Rosso* [1989] can be extended to account for the fractal dimension of individual streams. This yields estimates of the fractal dimension of the complete network a lot closer to 2. This agrees with our earlier work [*Tarboton et al.*, 1988] that used direct methods to estimate $D = 2$. In retrospect, it may be obvious that D should be 2, since in the limit of small resolution the network must fill the area it drains. River networks are examples of space-filling Peano curves as suggested by *Mandelbrot* [1977].

Assuming Horton's length and bifurcation laws hold exactly, *La Barbera and Rosso* [1989] derive:

$$D = \frac{\log R_b}{\log R_l} \tag{1}$$

The derivation of (1) by *La Barbera and Rosso* [1989] and also by *Tarboton et al.* [1988] is based on the measure of the total length of streams in the network being

$$L_T \sim \varepsilon^{1 - (\log R_b/\log R_l)} \tag{2}$$

where ε is the resolution with which first-order streams are detected. Comparing this with the relation suggested by *Mandelbrot* [1977]

$$L \sim r^{1-D} \tag{3}$$

where r is the linear resolution, (1) is obtained. Implicit in this comparison is the assumption that individual streams are linear measures and the sum of them, represented by (2), is a counting of linear lengths. This counting can be represented

$$L_T = N\varepsilon \tag{4}$$

where N is the number of measures ε the network is composed of. *La Barbera and Rosso* [1989] discuss the empirical mainstream length, basin area relationship

$$L_m \sim A^{\alpha} \tag{5}$$

Several empirical studies [see *Eagleson*, 1970, Figure 16-6] yield estimated $\alpha = 0.57$. This leads *Mandelbrot* [1977] and *Hjelmfelt* [1988] to suggest that individual rivers have fractal dimension

$$D_l = 2\alpha \simeq 1.14 \tag{6}$$

The stream length exceedance probability data of *Tarboton et al.* [1988, Figure 5] also supports this.

If the individual streams are themselves fractal with dimension D_l, then (3) cannot be compared to (2). Instead we have from (4) the equivalent count of number of fractal measures

$$N = \frac{L_T}{\varepsilon} \sim \varepsilon^{(-\log R_b)/(\log R_l)} \tag{7}$$

The fractal measure ε, with dimension D_l, is related to a linear measure r by

$$\varepsilon \sim r^{D_l} \tag{8}$$

This in (7) gives

$$N \sim r^{-D_l(\log R_b)/(\log R_l)} \tag{9}$$

This can be compared with the relationship given by *Mandelbrot* [1977]

$$N \sim r^{-D} \tag{10}$$

to get the fractal dimension of the whole network

$$D = D_l \frac{\log R_b}{\log R_l} \tag{11}$$

This is the extension to the result of *La Barbera and Rosso* [1989] that gives fractal dimension of the whole network in terms of Horton's laws and the fractal dimension of individual streams. With $\log R_b/\log R_l = 1.7$, and $D_l = 1.14$, this gives $D = 1.94$, hardly different from 2. Equation (11) gives

$$\frac{\log R_b}{\log R_l} = \frac{D}{D_l} \tag{12}$$

Now if the network is space filling with $D = 2$, and $D_l = 1.14$, then (12) gives the apparent D from (1) as $\log R_b/\log R_l = 1.75$.

We believe that this explains the observation [*La Barbera*

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and Rosso, 1989] that log R_b/log R_l is less than 2. The interpretation [*La Barbera and Rosso*, 1989] in terms of drainage density decreasing with increasing area is misleading. The quoted empirical relationship between drainage density and area is from many different basins, not within a single basin or river network. The phenomena may also be scale related due to the fact that when small basins are studied, higher-resolution maps are generally used. Drainage density gives a fundamental length scale. It is inversely proportional to mean link or first-order stream length [*Shreve*, 1967] and should not be related to scaling properties. The drainage density, or equivalently mean link length, defines the lower bound scale to the range of scales over which the network as a whole obeys Horton's laws and can be characterized by a fractal dimension.

At scales above this scale, the fact that the network drains the whole river basin constrains the network to be space filling, with fractal dimension $D = 2$. Putting $D = 2$ in (11) gives

$$R_b = R_l^{2/D_l} = R_l^{1.75} \tag{13}$$

for $D_l = 1.14$.

This generalizes the relationship between R_b and R_l suggested by *Tarboton et al.* [1988] to the case where the individual streams are also fractal.

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